Complex Variables

Chapter 4. Integrals

Section 4.50. Cauchy Integral Formula—Proofs of Theorems

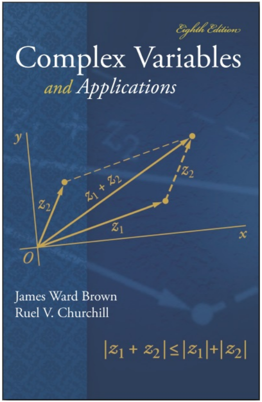

Table of contents

Theorem 4.50.A

Theorem 4.50.A. Cauchy Integral Formula.
Let f be analytic everywhere inside and on simple closed contour C, parameterized in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}.$$

Proof. Let C_ρ denote the positively oriented circle $|z - z_0| = \rho$, where ρ is small enough the C_ρ is interior to C (which can be done since C is a closed set and so $\mathbb{C} \setminus C$ is open with z_0 as an interior point of the open set $\mathbb{C} \setminus C$; see Figure 66).

Theorem 4.50.A

Theorem 4.50.A. Cauchy Integral Formula.
Let f be analytic everywhere inside and on simple closed contour C, parameterized in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}.$$

Proof. Let C_ρ denote the positively oriented circle $|z - z_0| = \rho$, where ρ is small enough the C_ρ is interior to C (which can be done since C is a closed set and so $\mathbb{C} \setminus C$ is open with z_0 as an interior point of the open set $\mathbb{C} \setminus C$; see Figure 66).

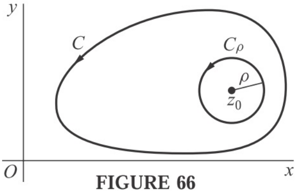

FIGURE 66

Theorem 4.50.A

Theorem 4.50.A. Cauchy Integral Formula.
Let f be analytic everywhere inside and on simple closed contour C, parameterized in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}.$$

Proof. Let C_ρ denote the positively oriented circle $|z - z_0| = \rho$, where ρ is small enough the C_ρ is interior to C (which can be done since C is a closed set and so $\mathbb{C} \setminus C$ is open with z_0 as an interior point of the open set $\mathbb{C} \setminus C$; see Figure 66).

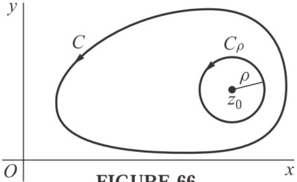

FIGURE 66

Theorem 4.50.A (continued 1)

Proof (continued). The function $f(z)/(z-z_0)$ is analytic inside and on C except at z_0. So by the Principle of Deformation (Corollary 4.49.B), $\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}$. So
$\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$. Next,
$\displaystyle\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i$ by Exercise 42.10(b), so

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i\,f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \qquad (4)$$

Since f is analytic, then it is continuous at z_0 and so for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|z-z_0| < \delta$ then $|f(z)-f(z_0)| < \varepsilon/(2\pi)$.

Theorem 4.50.A (continued 1)

Proof (continued). The function $f(z)/(z-z_0)$ is analytic inside and on C except at z_0. So by the Principle of Deformation (Corollary 4.49.B), $\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}$. So $\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$. Next, $\displaystyle\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i$ by Exercise 42.10(b), so

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i\, f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \qquad (4)$$

Since f is analytic, then it is continuous at z_0 and so for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|z-z_0| < \delta$ then $|f(z)-f(z_0)| < \varepsilon/(2\pi)$. The only restriction on ρ above is that C_ρ is interior to C. Let $\rho' = \min\{\rho, \delta/2\}$. Then $C_{\rho'}$ is interior to C and so the equations above involving C_ρ also hold for $C_{\rho'}$.

Theorem 4.50.A (continued 1)

Proof (continued). The function $f(z)/(z-z_0)$ is analytic inside and on C except at z_0. So by the Principle of Deformation (Corollary 4.49.B), $\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}$. So $\displaystyle\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho}\frac{dz}{z-z_0} = \int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}\,dz$. Next, $\displaystyle\int_{C_\rho}\frac{dz}{z-z_0} = 2\pi i$ by Exercise 42.10(b), so

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i\, f(z_0) = \int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}\,dz. \qquad (4)$$

Since f is analytic, then it is continuous at z_0 and so for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|z-z_0| < \delta$ then $|f(z)-f(z_0)| < \varepsilon/(2\pi)$. The only restriction on ρ above is that C_ρ is interior to C. Let $\rho' = \min\{\rho, \delta/2\}$. Then $C_{\rho'}$ is interior to C and so the equations above involving C_ρ also hold for $C_{\rho'}$.

Theorem 4.50.A (continued 2)

Proof (continued). Then for z on $C_{\rho'}$ we have $|z - z_0| = \rho' \leq \delta/2 < \delta$ and so $|f(z) - f(z_0)| < \varepsilon/(2\pi)$; also the length of $C_{\rho'}$ is $2\pi\rho'$ and so by Theorem 4.43.A,

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \left(\frac{\varepsilon/(2\pi)}{\rho'} \right) (2\pi\rho') = \varepsilon.$$

So by equation (4),

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| = \left| \int_C \frac{f(z)\, dz}{z - z_0} - 2\pi i f(z_0) \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then the quantity $\displaystyle\int_C \frac{f(z)\, dz}{z - z_0} - 2\pi i\, f(z_0)$ must be 0, and the result follows. □

Theorem 4.50.A (continued 2)

Proof (continued). Then for z on $C_{\rho'}$ we have $|z - z_0| = \rho' \leq \delta/2 < \delta$ and so $|f(z) - f(z_0)| < \varepsilon/(2\pi)$; also the length of $C_{\rho'}$ is $2\pi\rho'$ and so by Theorem 4.43.A,

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \left(\frac{\varepsilon/(2\pi)}{\rho'} \right) (2\pi\rho') = \varepsilon.$$

So by equation (4),

$$\left| \int_{C_{\rho'}} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| = \left| \int_C \frac{f(z)\, dz}{z - z_0} - 2\pi i f(z_0) \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then the quantity $\displaystyle\int_C \frac{f(z)\, dz}{z - z_0} - 2\pi i\, f(z_0)$ must be 0, and the result follows. □