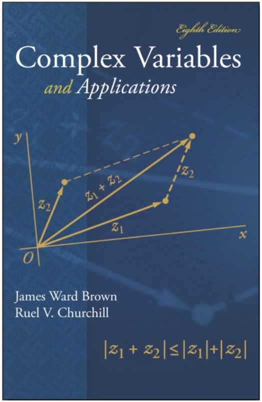
Complex Variables

Chapter 4. Integrals

Section 4.51. An Extension of the Cauchy Integral Formula—Proofs of Theorems



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$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds.$$

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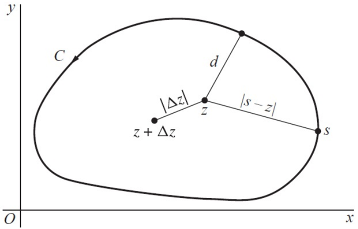

FIGURE 67

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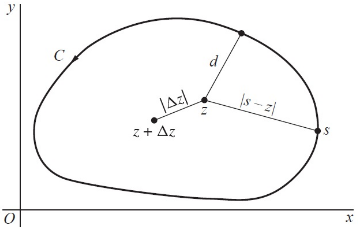

FIGURE 67

Lemma 4.51.A (continued 1)

Proof (continued). By the Cauchy Integral Formula (Theorem 4.50.A), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(s)\,ds}{s-z}$, so

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{\Delta z}\left(\frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-(z+\Delta z)} - \int_C \frac{f(s)\,ds}{s-z}\right)$$

$$= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)}.$$

Now

$$\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2},$$

so

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

Lemma 4.51.A (continued 2)

Proof (continued).

$$= \frac{1}{2\pi i} \int_C \left(\frac{1}{(s - z - \Delta z)(s - z)} - \frac{1}{(s - z)^2} \right) f(s)\, ds$$

$$= \frac{1}{2\pi i} \int_C \frac{\Delta z f(s)\, ds}{(s - z - \Delta z)(s - z)^2}. \qquad (*)$$

Next, let M denote the maximum value of $|f(s)|$ on C (which exists since $|f(s)|$ is continuous and C is compact) and observe that since $|s - z| > d$ (by the choice of d as a minimum distance) and $|\Delta z| < d$ (by the choice of Δz) then

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \text{ by Corollary 1.4.1}$$

$$\geq |s - z| - |\Delta z| \geq d - |\Delta z| > 0.$$

Lemma 4.51.A (continued 2)

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$$= \frac{1}{2\pi i}\int_C \left(\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right) f(s)\,ds$$

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$$|s-z-\Delta z| = |(s-z)-\Delta z| \geq ||s-z|-|\Delta z|| \text{ by Corollary 1.4.1}$$

$$\geq |s-z|-|\Delta z| \geq d-|\Delta z| > 0.$$

Lemma 4.51.A (continued 3)

Proof (continued). Thus by Theorem 4.43.A

$$\left|\int_C \frac{\Delta z f(s)\,ds}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{|\Delta z| M}{(d-|\Delta z|)d^2} L$$

where L is the length of C. So from $(*)$, this implies

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}\right| = \frac{1}{2\pi}\left|\int_C \frac{\Delta z f(s)\,ds}{(s-z-\Delta z)(s-z)^2}\right|$$

$$\le \frac{|\Delta z| M}{2\pi(d-|\Delta z|)d^2} L$$

and so as $\Delta z \to 0$ we see that $\dfrac{|\Delta z| M}{2\pi(d-|\Delta z|)d^2} L \to 0$. Hence,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds.$$

Therefore, $f'(z)$ exists and has the claimed value. □

Lemma 4.51.A (continued 3)

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$$\left| \int_C \frac{\Delta z f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right| \leq \frac{|\Delta z| M}{(d - |\Delta z|) d^2} L$$

where L is the length of C. So from $(*)$, this implies

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - z)^2} \right| = \frac{1}{2\pi} \left| \int_C \frac{\Delta z f(s)\, ds}{(s - z - \Delta z)(s - z)^2} \right|$$

$$\leq \frac{|\Delta z| M}{2\pi (d - |\Delta z|) d^2} L$$

and so as $\Delta z \to 0$ we see that $\dfrac{|\Delta z| M}{2\pi (d - |\Delta z|) d^2} L \to 0$. Hence,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2}\, ds.$$

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