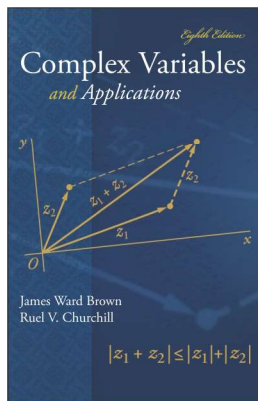


Complex Variables

Chapter 4. Integrals

Section 4.52. Some Consequences of the Extension—Proofs of Theorems



Corollary 4.52.A

Corollary 4.52.A

Corollary 4.52.A. If a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is analytic at a point $z_0 = x_0 + iy_0$ then the component functions u and v have continuous partial derivatives of all orders at the point.

“Proof.” We have by Theorem 2.21.A that $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. Since all orders of derivatives $f^{(n)}(z_0)$ exist by Theorem 4.52.1 then all partial derivatives of u and v with respect to x exist. By the Cauchy-Riemann equations (also Theorem 2.21.A), $u_x = v_y$ and $u_y = -v_x$. From this, one can show that all partials with respect to y and all mixed partials of u and v exist (some details here would be welcome!). Hence, u and v have continuous partial derivatives of all orders at (x_0, y_0) . \square

Theorem 4.52.1

Theorem 4.52.1. If a function f is analytic at a given point, then its derivatives of all orders are analytic at that point too.

Proof. Let f be analytic at a point z_0 . Then by the definition of analytic (namely, differentiable on a neighborhood of z_0), there is $\varepsilon > 0$ such that f is analytic at all points of the disk $|z - z_0| < \varepsilon$. Let C_0 be the positively oriented circle $|z - z_0| = \varepsilon/2$. Then f is analytic inside and on C_0 and for $n \in \mathbb{N}$, by the Extended Cauchy's Formula (Theorem 4.51.A), we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_0} \frac{f(s) ds}{(s - z)^{n+1}}.$$

This holds for all z where $|z - z_0| < \varepsilon/2$. So $f^{(n)}(z)$ exists for all such z and hence $f^{(n-1)}(z)$ is differentiable for all $|z - z_0| < \varepsilon/2$; that is, $f^{(n-1)}(z)$ is analytic at z_0 . Since $n \in \mathbb{N}$ is arbitrary, we have that all orders of derivative of f are analytic at z_0 , as claimed. \square

Theorem 4.52.2. Morera's Theorem

Theorem 4.52.1

Theorem 4.52.2. Morera's Theorem.

Let f be continuous on a domain D . If $\int_C f(z) dz = 0$ for every closed contour C in D , then f is analytic throughout D .

Proof. We have by Theorem 4.44.A that $f(z)$ has an antiderivative $F(z)$ throughout D (the (c) implies (a) part). So for each $z \in D$, $F'(z) = f(z)$ and so F is differentiable on domain D and hence F is analytic on D . By Theorem 4.52.1, $f = F'$ is analytic on D . \square

Theorem 4.52.3

Theorem 4.52.3. Cauchy's Inequality.

Suppose that function f is analytic inside and on a positively oriented circle C_R centered at z_0 with radius R . If M_R is the maximum value of $|f(z)|$ on C_R , then $|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}$ for $n \in \mathbb{N}$.

Proof. By the Extended Cauchy Formula (Theorem 4.51.A),

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}.$$

Notice that for $z \in C_R$ we have $|z - z_0| = R$, and the length of C_R is $L = 2\pi R$. So

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi} \left| \int_C \frac{f(z) dz}{(z - z_0)^{n+1}} \right| \\ &\leq \frac{n!}{2\pi} \left(\frac{M_R}{R^{n+1}} \right) (2\pi R) \text{ by Theorem 4.43.A} \\ &= \frac{n!M_R}{R^n}. \end{aligned}$$

□