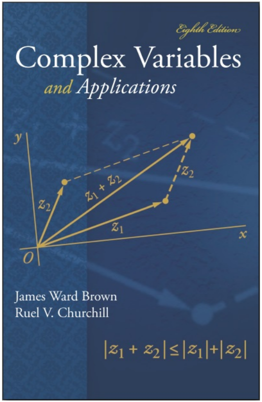
Complex Variables

Chapter 4. Integrals

Section 4.52. Some Consequences of the Extension—Proofs of Theorems



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$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)^{n+1}}.$$

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Corollary 4.52.A

Corollary 4.52.A. If a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is analytic at a point $z_0 = x_0 + iy_0$ then the component functions u and v have continuous partial derivatives of all orders at the point.

"Proof." We have by Theorem 2.21.A that $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. Since all orders of derivatives $f^{(n)}(z_0)$ exist by Theorem 4.52.1 then all partial derivatives of u and v with respect to x exist.

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Theorem 4.52.1

Theorem 4.52.2. Morera's Theorem.
Let f be continuous on a domain D. If $\int_C f(z)\,dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Proof. We have by Theorem 4.44.A that $f(z)$ has an antiderivative $F(z)$ throughout D (the (c) implies (a) part). So for each $z \in D$, $F'(z) = f(z)$ and so F is differentiable on domain D and hence F is analytic on D. By Theorem 4.52.1, $f = F'$ is analytic on D. □

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Theorem 4.52.3

Theorem 4.52.3. Cauchy's Inequality.
Suppose that function f is analytic inside and on a positively oriented circle C_R centered at z_0 with radius R. If M_R is the maximum value of $|f(z)|$ on C_R, then $|f^{(n)}(z_0)| \leq \dfrac{n!M_R}{R^n}$ for $n \in \mathbb{N}$.

Proof. By the Extended Cauchy Formula (Theorem 4.51.A), $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_C \dfrac{f(z)\,dz}{(z-z_0)^{n+1}}$. Notice that for $z \in C_R$ we have $|z - z_0| = R$, and the length of C_R is $L = 2\pi R$.

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