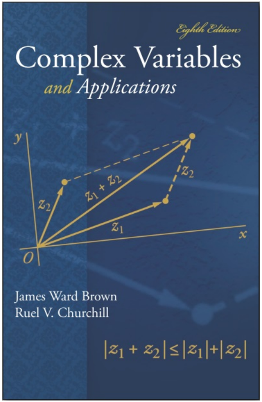
Complex Variables

Chapter 4. Integrals

Section 4.53. Liouville's Theorem and the Fundamental Theorem of Algebra—Proofs of Theorems



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If a function f is entire and bounded in the whole complex plane, then f is constant throughout the entire complex plane.

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Theorem 4.53.2

Theorem 4.53.2. The Fundamental Theorem of Algebra.
Any complex polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, of degree $n \geq 1$ has at least one zero. That is, there exists at least one point $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Proof. ASSUME no such z_0 exists and that $P(z)$ is nonzero throughout $\mathbb{C}$. Then by Lemma 2.24.A, the function $1/P(z)$ is analytic throughout $\mathbb{C}$; that is, $1/P(z)$ is an entire function.

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We claim that $1/P(z)$ is bounded. Notice that

$$P(z) = \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z} + a_n\right) z^n.$$

Since

$$\lim_{z\to\infty}\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right) = 0,$$

then for $\varepsilon = |a_n|/2$ there is $R > 0$ such that for all $|z| > R$ we have...

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Theorem 4.53.2 (continued 1)

Proof (continued). ...

$$\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right| < \frac{|a_n|}{2} = \varepsilon.$$

So for $|z| > R$,

$$\begin{aligned}
&\left|\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right) + a_n\right| \\
&\geq \left|\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right| - |a_n|\right| \text{ by Corollary 1.4.1} \\
&> |a_n|/2.
\end{aligned}$$

So

$$\begin{aligned}
|P(z)| &= \left|\left\{\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right) + a_n\right\} z^n\right| \\
&= \left|\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right) + a_n\right| |z|^n \\
&> |a_n||z|^n/2 > |a_n|R^n/2 \text{ for } |z| > R.
\end{aligned}$$

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$$\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right| < \frac{|a_n|}{2} = \varepsilon.$$

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$$\begin{aligned} &\left|\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right) + a_n\right| \\ &\geq \left|\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}\right| - |a_n|\right| \text{ by Corollary 1.4.1} \\ &> |a_n|/2. \end{aligned}$$

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Proof (continued). So $|1/P(z)| < 2/(|a_n|R^n)$ for $|z| > R$. Now $1/P(z)$ is continuous by assumption and so by Theorem 2.18.3, $|1/P(z)|$ is bounded, by say M, on the closed and bounded set $|z| \leq R$. Therefore

$$\left|\frac{1}{P(z)}\right| \leq \begin{cases} |a_n|R^n/2 & \text{for } |z| > R \\ M & \text{for } |z| \leq R \end{cases}$$

and $1/P(z)$ is a bounded entire function.

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But Liouville's Theorem then implies that $1/P(z)$ is constant, a CONTRADICTION. So the assumption that $P(z)$ is nonzero throughout $\mathbb{C}$ is false and there must be some $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. □

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