Complex Variables

Chapter 4. Integrals

Section 4.54. Maximum Modulus Principle—Proofs of Theorems

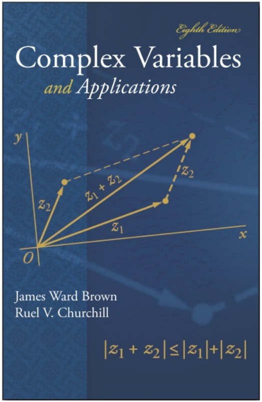

Table of contents

Lemma 4.54.A

Lemma 4.54.A. Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Proof. Let $z_1 \neq z_0$ be in the ε-neighborhood of z_0. Let $\rho = |z_1 - z_0|$. Let C_ρ be the positively oriented circle $|z - z_0| = \rho$. Then f is analytic on and inside C_ρ and so by the Cauchy Integral Formula (Theorem 4.50.A),
$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)\, dz}{z - z_0}$.

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$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta}) - z_0} i\rho e^{i\theta}\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta. \quad (2)$$

We then have from (2) that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta \text{ by Lemma 4.43.A.} \quad (3)$$

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Proof (continued). On the other hand, by hypothesis, $|f(z_0 + \rho e^{i\theta})| \leq |f(z_0)|$ for $\theta \in [0, 2\pi]$ so that $\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \int_0^{2\pi} |f(z_0)| \, d\theta = 2\pi |f(z_0)|$, or

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Combining equations (3) and (5) gives $|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta$, or $\int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|) \, d\theta = 0$. Now the integrand is nonnegative by hypothesis and is a continuous function of θ on $[0, 2\pi]$. If the integral of a continuous real-valued nonnegative function over some interval is 0 then the function must be identically 0 (yes, we could use a reference for this).

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Theorem 4.54.C

Theorem 4.54.C. The Maximum Modulus Theorem.
If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

Proof. Let f be analytic and nonconstant on D. ASSUME $|f(z)|$ has a maximum on D of $|f(z_0)|$ for some $z_0 \in D$.

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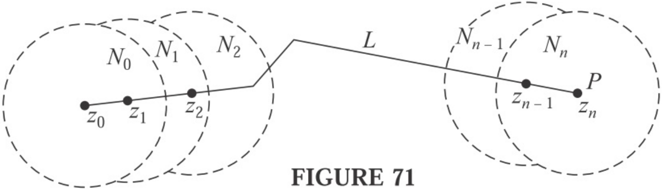

FIGURE 71

Since $|f(z_0)|$ is a maximum of $|f(z)|$ on D, then it is a maximum on N_0 and $z_0 \in N_0$, so by Lemma 4.54.A, f is constant on N_0. In particular, $f(z_1) = f(z_0)$. So $|f(z_1)|$ is a maximum of $|f(z)|$ on N_1 and $z_1 \in N_1$, so by Lemma 4.54.A, f is a constant on N_1. Inductively, f is constant on $N_0 \cup N_1 \cup \cdots \cup N_n$ and so $f(z_0) = f(P)$. Since P is an arbitrary point of D, then f is constant on D, a CONTRADICTION. So the assumption that $|f(z)|$ has a maximum on D is false, and $|f(z)|$ has no maximum on D, as claimed. □

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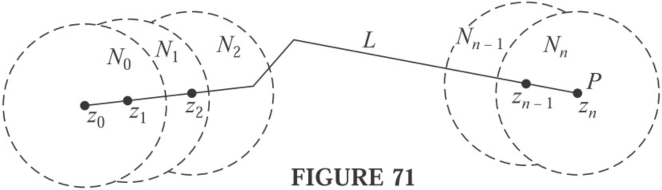

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Theorem 4.54.D

Theorem 4.54.D. Maximum Modulus Theorem, Alternative Version. Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ on R, which is always reached (by Theorem 2.18.3) occurs somewhere on the boundary of R and never in the interior.

Proof. Let M be the maximum of $|f(z)|$ on R, so that $|f(z)| \leq M$ for all $z \in R$. If f is constant, then $|f(z)| = M$ for all $z \in R$ and so the maximum is attained on the boundary.

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Theorem 4.54.E

Theorem 4.54.E. Let f be continuous on a closed bounded region R, and analytic and not constant on the interior of R. For $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, function $u(x,y)$ attains its maximum value in R on the boundary of R and not in the interior.

Proof. Let $g(z) = e^{f(z)}$. Then g is continuous on R and analytic in the interior of R (by Theorem 2.18.1 and Lemma 2.24.B). Next,

$$|g(z)| = |e^{f(z)}| = |e^{u(x,y)+iv(x,y)}| = |e^{u(x,y)}||e^{iv(x,y)}| = e^{u(x,y)}.$$

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So by Corollary 4.54.D, $|g(z)| = e^{u(x,y)}$ attains the maximum on the boundary of R. Since e^x is an increasing function of real variable x, then $u(x,y)$ attains its maximum at the same point on the boundary of R. Since f is not a constant then the maximum $u(x,y)$ (and hence $|g(z)|$) cannot occur at an interior point also by Corollary 4.54.D. □

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