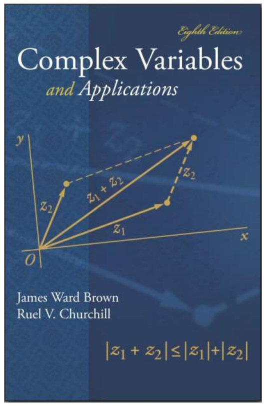
Complex Variables

Chapter 5. Series

Section 5.55. Convergence of Sequences—Proofs of Theorems



Theorem 5.55.A

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Proof. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Let $\varepsilon > 0$. Then for some $n_1, n_2 \in \mathbb{N}$, $|x_n - x| < \varepsilon/2$ whenever $n > n_1$ and $|y_n - y| < \varepsilon/2$ whenever $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then for $n > n_0$ we have

$$|z_n - z| = |(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |i(y_n - y)|$$

$$= |x_n - x| + |y_n - y| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{n\to\infty} z_n = z$.

Theorem 5.55.A (continued)

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Proof (continued). Conversely, suppose that $\lim_{n\to\infty} z_n = z$. Let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that $|(x_n + iy_n) - (x + iy)| < \varepsilon$ whenever $n > n_0$. But

$$|x_n - x| \le \sqrt{(x_n - x)^+(y_n - y)^2} = |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and

$$|y_n - y| \le \sqrt{(x_n - x)^+(y_n - y)^2} = |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|,$$

so both $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ whenever $n > n_0$. That is $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. □