Complex Variables

Chapter 5. Series

Section 5.56. Convergence of Series—Proofs of Theorems

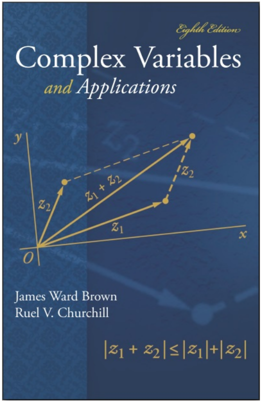

Table of contents

Theorem 5.56.A

Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Proof. Let $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Then

$$S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N}(x_n + iy_n) = \sum_{n=1}^{N} x_n + i\sum_{n=1}^{N} y_n = X_N + iY_N.$$

Theorem 5.56.A

Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Proof. Let $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Then

$$S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n = X_N + iY_N.$$

So $\sum_{n=1}^{\infty} z_n = S$ if and only if $\lim_{n\to\infty} S_N = S$; that is, if and only if $\lim_{n\to\infty}(X_n + iY_n) = S$. Now by Theorem 5.55.A, $\lim_{n\to\infty}(X_n + iY_n) = \lim_{n\to\infty} X_n + i \lim_{n\to\infty} Y_n = X + iY$. So $\sum_{n=1}^{\infty} z_n = S$ if and only if $S = X + iY$, as claimed. □

Theorem 5.56.A

Theorem 5.56.A. Suppose that $z_n = x_n + iy_n$ and $S = X + iY$. Then $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Proof. Let $X_N = \sum_{n=1}^{N} x_n$ and $Y_N = \sum_{n=1}^{N} y_n$. Then

$$S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} (x_n + iy_n) = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n = X_N + iY_N.$$

So $\sum_{n=1}^{\infty} z_n = S$ if and only if $\lim_{n\to\infty} S_N = S$; that is, if and only if $\lim_{n\to\infty}(X_n + iY_n) = S$. Now by Theorem 5.55.A, $\lim_{n\to\infty}(X_n + iY_n) = \lim_{n\to\infty} X_n + i \lim_{n\to\infty} Y_n = X + iY$. So $\sum_{n=1}^{\infty} z_n = S$ if and only if $S = X + iY$, as claimed. $\square$

Corollary 5.56.1

Corollary 5.56.1. Test for Divergence
If a series of complex numbers converges, then the *n*th term converges to zero as n tends to infinity. That is, if z_n does not converge to 0 then $\sum_{n=1}^{\infty} z_n$ diverges.

Proof. Let $\sum_{n=1}^{\infty} z_n$ converge. With $z_n = x_n + iy_n$, Theorem 5.56.A implies that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge.

Corollary 5.56.1

Corollary 5.56.1. Test for Divergence
If a series of complex numbers converges, then the *n*th term converges to zero as n tends to infinity. That is, if z_n does not converge to 0 then $\sum_{n=1}^{\infty} z_n$ diverges.

Proof. Let $\sum_{n=1}^{\infty} z_n$ converge. With $z_n = x_n + iy_n$, Theorem 5.56.A implies that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge. By the Test for Divergence from Calculus 2 (see Theorem 7 of my online notes on 10.2. Infinite Series), we have that x_n converges to 0 and y_n converges to 0. So by Theorem 5.55.A,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + i0 = 0.$$

□

Corollary 5.56.1

Corollary 5.56.1. Test for Divergence
If a series of complex numbers converges, then the nth term converges to zero as n tends to infinity. That is, if z_n does not converge to 0 then $\sum_{n=1}^{\infty} z_n$ diverges.

Proof. Let $\sum_{n=1}^{\infty} z_n$ converge. With $z_n = x_n + iy_n$, Theorem 5.56.A implies that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ both converge. By the Test for Divergence from Calculus 2 (see Theorem 7 of my online notes on 10.2. Infinite Series), we have that x_n converges to 0 and y_n converges to 0. So by Theorem 5.55.A,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + i0 = 0.$$

□

Corollary 5.56.2

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \le \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \le \sqrt{x_n^2 + y_n^2} = |z_n|$.

Corollary 5.56.2

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$. So by the Direct Comparison Test for series of real numbers from Calculus 2 (see Theorem 10. The (Direct) Comparison Test in my online notes on 10.4. Comparison Tests), we see that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge.

So both the series of real numbers $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge absolutely.

Corollary 5.56.2

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof. Suppose series $\sum_{n=0}^{\infty} z_n$ converges absolutely. With $z_n = x_n + iy_n$, we have $|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$ and $|y_n| = \sqrt{y_n^2} \leq \sqrt{x_n^2 + y_n^2} = |z_n|$. So by the Direct Comparison Test for series of real numbers from Calculus 2 (see Theorem 10. The (Direct) Comparison Test in my online notes on 10.4. Comparison Tests), we see that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge.

So both the series of real numbers $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge absolutely.

Corollary 5.56.2 (continued)

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued. Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge.

Corollary 5.56.2 (continued)

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued. Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge. Therefore, by Theorem 5.56.A, the series $\sum_{n=0}^{\infty} z_n$ converges. □

Corollary 5.56.2 (continued)

Corollary 5.56.2. The absolute convergence of a series of complex numbers implies the convergence of that series.

Proof (continued. Since the absolute convergence of a series of real numbers implies its convergence (see Theorem 16. The Absolute Convergence Theorem in my online Calculus 2 notes on 10.6. Alternating Series, Absolute and Conditional Convergence), then the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge. Therefore, by Theorem 5.56.A, the series $\sum_{n=0}^{\infty} z_n$ converges. □