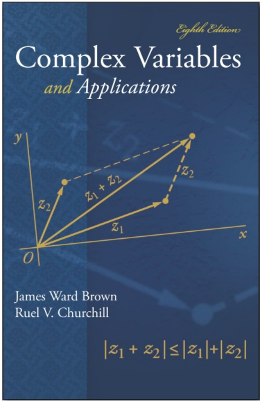
Complex Variables

Chapter 5. Series

Section 5.58. Proof of Taylor's Theorem—Proofs of Theorems



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Theorem 5.57.A. Taylor's Theorem. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ (that is, $f'(z)$ is defined for each $|z - z_0| < R_0$), centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $|z - z_0| < R_0$ where $a_n = f^{(n)}(z_0)/n!$ for $n = 0, 1, 2, \ldots$. That is, the series converges to $f(z)$ for each z in the stated disk.

Proof. Let $z_0 = 0$, $|z| = r$ and let C_0 denote the positively oriented circle $|z| = r_0$ where $r < r_0 < R_0$. Since f is analytic inside and on C_0 by hypothesis and since the point z is interior to C_0, then by the Cauchy Integral Formula (Theorem 4.50.A),

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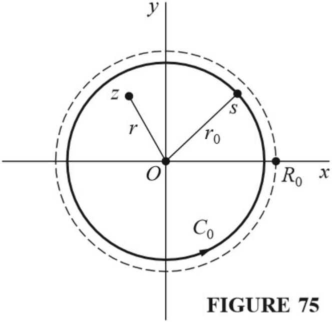

FIGURE 75

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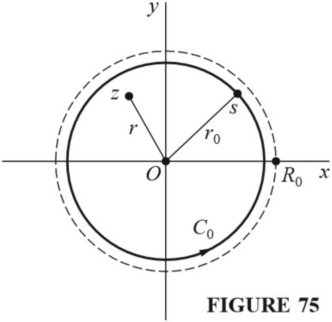

FIGURE 75

Theorem 5.57.A (continued 1)

Proof (continued).

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\, ds}{s - z} \qquad (1)$$

Notice that $\dfrac{1}{s-z} = \dfrac{1}{s}\dfrac{1}{1-z/s}$, and as seen in the last example of Section 56, $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$ for any $z \neq 1$. So we have

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(s-z)s^N}.$$

Multiplying both sides by $f(s)$ and then integrating each side with respect to s around C_0 gives,

$$\int_{C_0} \frac{f(s)\, ds}{s-z} = \sum_{n=0}^{N-1} \left(\int_{C_0} \frac{f(s)\, ds}{s^{n+1}} \right) z^n + z^N \int_{C_0} \frac{f(s)\, ds}{(s-z)S^N}. \qquad (*)$$

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Theorem 5.57.A (continued 2)

Proof (continued). By the Extended Cauchy Formula (Theorem 4.51.A) (with $z_0 = 0$), $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_{C_0} \frac{f(s)\,ds}{s^{n+1}} = f^{(n)}(0)$ for $n = 0, 1, 2, \ldots$. So from $(*)$ we get

$$\frac{1}{2\pi i}\int_{C_0} \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\sum_{n=0}^{N-1}\left(\int_{C_0}\frac{f(s)\,ds}{s^{n+1}}\right) z^n + \frac{z^N}{2\pi i}\int_{C_0}\frac{f(s)\,ds}{(s-z)S^N},$$

and so from (1) and the Extended Cauchy Formula,

$$f(z) = \sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}z^n + \rho_N(z), \qquad (5)$$

where $\rho_N(z) = \dfrac{z^N}{2\pi i}\displaystyle\int_{C_0}\frac{f(s)\,ds}{(s-z)z^N}$.

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Theorem 5.57.A (continued 3)

Proof (continued). Now we have $|z| = r$ and C_0 has radius r_0 where $r_0 > r$. So for s on C_0 we have by Corollary 1.4.1 that $|s - z| \geq ||s| - |z|| = r_0 - r$. With $M = \max_{s \in C_0} |f(s)|$, we now have

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \frac{M}{(r_0 - r) r_0^N} 2\pi r_0 = \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N .$$

Since $r/r_0 < 1$, then $\lim_{N \to \infty} (r/r_0)^N = 0$, and so from (5),

$$f(z) = \lim_{N \to \infty} \left(\sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n \right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n .$$

So the result holds for $z_0 = 0$.

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So the result holds for $z_0 = 0$.

Theorem 5.57.A (continued 4)

Theorem 5.57.A. Taylor's Theorem. Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ (that is, $f'(z)$ is defined for each $|z - z_0| < R_0$), centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R_0$ where $a_n = f^{(n)}(z_0)/n!$ for $n = 0, 1, 2, \ldots$.

Proof (continued). Now suppose that f is analytic on $|z - z_0| < R_0$ and let $g(z) = f(z + z_0)$. Then g is analytic on $|z| < R_0$ and by the above argument, $g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$ for $|z| < R_0$. That is,

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \text{ for } |z| < R_0, \text{ and replacing } z \text{ with } z - z_0,$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \text{ for } |z - z_0| < R_0, \text{ as claimed.}$$

□