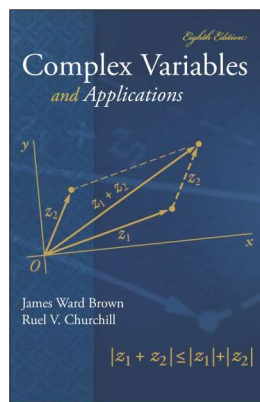


# Complex Variables

## Chapter 5. Series

Section 5.63. Absolute and Uniform Convergence of Power Series—Proofs of Theorems



### Theorem 5.63.1

## Theorem 5.63.1

**Proof (continued).** Then

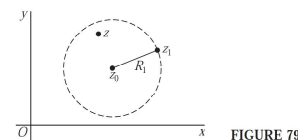
$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \frac{|z - z_0|^n}{|z_1 - z_0|^n} \leq M\rho^n \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Now  $\sum_{n=0}^{\infty} M\rho^n$  is a geometric series (of real numbers) with ratio  $\rho < 1$  and so it converges (to  $M/(1 - \rho)$ ). So by the Direct Comparison Test (see my online Calculus 2 notes on [10.4, Comparison Tests](#)) we know that the series of real numbers  $\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$  converges. That is,  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges absolutely for all  $z$  satisfying  $|z - z_0| < R_1$ , as claimed.  $\square$

### Theorem 5.63.1

## Theorem 5.63.1

**Theorem 5.63.1.** If a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges when  $z = z_1$  (where  $z_1 \neq z_0$ ), then the power series is absolutely convergent at each point  $z$  in the disk  $|z - z_0| < R_1$  where  $R_1 = |z_1 - z_0|$  (see Figure 79).



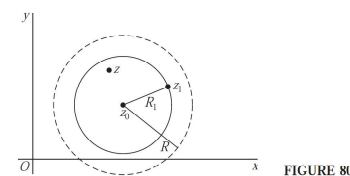
**Proof.** Suppose  $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$  converges for some  $z_1 \neq z_0$ . By Corollary 5.56.A, the series is bounded so there is some positive  $M \in \mathbb{R}$  such that  $|a_n(z_1 - z_0)^n| \leq M$  for  $n \in \mathbb{N} \cup \{0\}$ . For  $z \in \mathbb{C}$  with  $|z - z_0| < R_1$ , let  $\rho = |z - z_0|/|z_1 - z_0|$  so that  $\rho < 1$ .

### Theorem 5.63.2

## Theorem 5.63.2

**Theorem 5.63.2.** If  $z_1$  is a point inside the circle of convergence  $|z - z_0| = R$  of a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ , then the series is uniformly convergent in every closed disk  $|z - z_0| < R_1$  where  $0 \leq R_1 < R$ .

**Proof.** Given a point  $z_1$  on the circle  $|z - z_0| = R_1$ , there are points inside the circle of convergence which are farther from  $z_0$  than  $z_1$  is, for which the series converges (see Figure 80).



So by Theorem 5.63.1, the series  $\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n|$  converges.

## Theorem 5.63.2 (continued 1)

**Proof (continued).** Let  $m, N \in \mathbb{N}$  where  $m > N$ . Then the convergence of the power series implies

$$\rho_N(z) = S(z) - S_N(z) = \sum_{n=N}^{\infty} a_n(z - z_0)^n = \lim_{m \rightarrow \infty} \sum_{n=N}^m a_n(z - z_0)^n$$

and the absolute convergence of the power series at  $z = z_1$  lets us define the real number

$$\sigma_N = \sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n| = \lim_{m \rightarrow \infty} \sum_{n=0}^m |a_n(z_1 - z_0)^n|.$$

By Exercise 5.56.3,

$$|\rho_N(z)| = \left| \lim_{m \rightarrow \infty} \sum_{n=N}^m a_n(z - z_0)^n \right| = \lim_{m \rightarrow \infty} \left| \sum_{n=N}^m a_n(z - z_0)^n \right|, \dots$$

## Theorem 5.63.2 (continued 2)

**Proof (continued).** ... and when  $|z - z_0| \leq |z_1 - z_0|$  we have

$$\begin{aligned} \left| \sum_{n=N}^m a_n(z - z_0)^n \right| &\leq \sum_{n=N}^m |a_n| |z - z_0|^n \text{ by the Triangle Inequality} \\ &\leq \sum_{n=N}^{\infty} |a_n| |z_1 - z_0|^n \text{ since } |z - z_0| \leq |z_1 - z_0| \\ &= \sum_{n=N}^{\infty} |a_n(z_1 - z_0)^n|. \end{aligned}$$

Consequently, when  $|z - z_0| \leq R_1$  we have

$$|\rho_N(z)| = \lim_{m \rightarrow \infty} \left| \sum_{n=N}^m a_n(z - z_0)^n \right| \leq \lim_{m \rightarrow \infty} \sum_{n=N}^m |a_n(z_1 - z_0)^n| = \sigma_N.$$

## Theorem 5.63.2 (continued 3)

**Theorem 5.63.2.** If  $z_1$  is a point inside the circle of convergence  $|z - z_0| = R$  of a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ , then the series is uniformly convergent in every closed disk  $|z - z_0| \leq R_1$  where  $0 \leq R_1 < R$ .

**Proof (continued).** Since the  $\sigma_N$  are the remainder of the convergent series  $\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n|$ , then they tend to zero as  $N$  tends to infinity (by Note 5.56.A). That is, for each  $\varepsilon > 0$  there is  $N_\varepsilon \in \mathbb{N}$  such that  $|\rho_N(z)| < \sigma_N < \varepsilon$  whenever  $N > N_\varepsilon$ . This holds for any  $z$  in the disk  $|z - z_0| \leq R_1$  and, therefore, the series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges uniformly on the closed disk  $|z - z_0| \leq R_1$ , as claimed.  $\square$