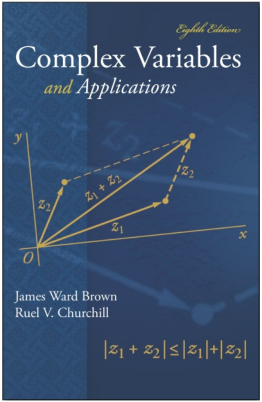
Complex Variables

Chapter 5. Series

Section 5.63. Absolute and Uniform Convergence of Power Series—Proofs of Theorems



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Theorem 5.63.1. If a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges when $z = z_1$ (where $z_1 \neq z_0$), then the power series is absolutely convergent at each point z in the disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$ (see Figure 79).

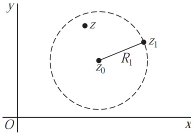

FIGURE 79

Proof. Suppose $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges for some $z_1 \neq z_0$. By Corollary 5.56.A, the series is bounded so there is some positive $M \in \mathbb{R}$ such that $|a_n(z_1 - z_0)^n| \leq M$ for $n \in \mathbb{N} \cup \{0\}$. For $z \in \mathbb{C}$ with $|z - z_0| < R_1$, let $\rho = |z - z_0|/|z_1 - z_0|$ so that $\rho < 1$.

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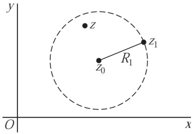

FIGURE 79

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Proof (continued). Then

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n|\frac{|z-z_0|^n}{|z_1-z_0|^n} \le M\rho^n \text{ for } n \in \mathbb{N}\cup\{0\}.$$

Now $\sum_{n=0}^{\infty} M\rho^n$ is a geometric series (of real numbers) with ratio $\rho < 1$ and so it converges (to $M/(1-\rho)$). So by the Direct Comparison Test (see my online Calculus 2 notes on 10.4, Comparison Tests) we know that the series of real numbers $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges. That is, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely for all z satisfying $|z-z_0| < R_1$, as claimed. □

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Theorem 5.63.2

Theorem 5.63.2. If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, then the series is uniformly convergent in every closed disk $|z - z_0| < R_1$ where $0 \leq R_1 < R$.

Proof. Given a point z_1 on the circle $|z - z_0| = R_1$, there are points inside the circle of convergence which are farther from z_0 than z_1 is, for which the series converges (see Figure 80).

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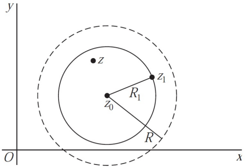

FIGURE 80

So by Theorem 5.63.1, the series $\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n|$ converges.

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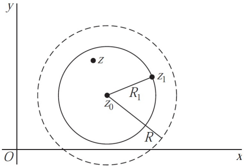

FIGURE 80

So by Theorem 5.63.1, the series $\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n|$ converges.

Theorem 5.63.2 (continued 1)

Proof (continued). Let $m, N \in \mathbb{N}$ where $m > N$. Then the convergence of the power series implies

$$\rho_N(z) = S(z) - S_N(z) = \sum_{n=N}^{\infty} a_n(z - z_0)^n = \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z - z_0)^n$$

and the absolute convergence of the power series at $z = z_1$ lets us define the real number

$$\sigma_N = \sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n| = \lim_{m\to\infty} \sum_{n=0}^{m} |a_n(z_1 - z_0)^n|.$$

By Exercise 5.56.3,

$$|\rho_N(z)| = \left| \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z - z_0)^n \right| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n(z - z_0)^n \right|, \ldots$$

Theorem 5.63.2 (continued 2)

Proof (continued). ... and when $|z - z_0| \leq |z_1 - z_0|$ we have

$$\begin{aligned} \left| \sum_{n=N}^{m} a_n (z - z_0)^n \right| &\leq \sum_{n=N}^{m} |a_n| |z - z_0|^n \text{ by the Triangle Inequality} \\ &\leq \sum_{n=N}^{\infty} |a_n| |z_1 - z_0|^n \text{ since } |z - z_0| \leq |z_1 - z_0| \\ &= \sum_{n=N}^{\infty} |a_n (z_1 - z_0)^n|. \end{aligned}$$

Consequently, when $|z - z_0| \leq R_1$ we have

$$|\rho_N(z)| = \lim_{m \to \infty} \left| \sum_{n=N}^{m} a_n (z - z_0)^n \right| \leq \lim_{m \to \infty} \sum_{n=N}^{m} |a_n (z_1 - z_0)^n| = \sigma_N.$$

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Theorem 5.63.2 (continued 3)

Theorem 5.63.2. If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, then the series is uniformly convergent in every closed disk $|z - z_0| < R_1$ where $0 \leq R_1 < R$.

Proof (continued). Since the σ_N are the remainder of the convergent series $\sum_{n=0}^{\infty} |a_n(z_1 - z_0)^n|$, then they tend to zero as N tends to infinity (by Note 5.56.A). That is, for each $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that $|\rho_N(z)| < \sigma_N < \varepsilon$ whenever $N > N_\varepsilon$. This holds for any z in the disk $|z - z_0| \leq R_1$ and, therefore, the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on the closed disk $|z - z_0| \leq R_1$, as claimed. □

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