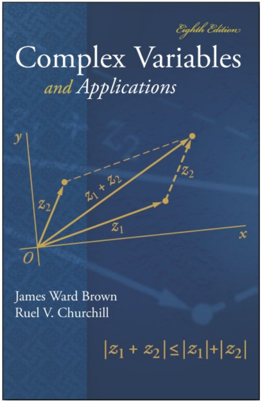
Complex Variables

Chapter 5. Series

Section 5.64. Continuity of Sums of Power Series—Proofs of Theorems



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Theorem 5.64.1. A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z-z_0| = R$.

Proof. Let $S_N(z) = \sum_{n=0}^{N-1} a_n(z-z_0)^n$ and consider the remainder function $\rho_N(z) = S(z) - S_N(z)$ for $|z-z_0| < R$. Then, because $S(z) = S_N(z) + \rho_N(z)$ for $|z-z_0| < R$, we have

$$|S(z) - S(z_1)| = |(S_N(z) + \rho_N(z)) - (S_N(z_1) + \rho_N(z_1))|$$

$$\leq |S_N(z) - S_N(z_1)| + |\rho_n(z)| + |\rho_N(z_1)| \text{ by the Triangle Inequality.} \quad (*)$$

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Proof. Let $S_N(z)=\sum_{n=0}^{N-1} a_n(z-z_0)^n$ and consider the remainder function $\rho_N(z)=S(z)-S_N(z)$ for $|z-z_0|<R$. Then, because $S(z)=S_N(z)+\rho_N(z)$ for $|z-z_0|<R$, we have

$$|S(z)-S(z_1)| = |(S_N(z)+\rho_N(z))-(S_N(z_1)+\rho_N(z_1))|$$

$$\leq |S_N(z)-S_N(z_1)|+|\rho_n(z)|+|\rho_N(z_1)| \text{ by the Triangle Inequality.} \quad (*)$$

Let $\varepsilon>0$. If z is any point in some closed disk $|z-z_0|\leq R_0$ where $|z_1-z_0|<R_0<R_1$, then by the uniform convergence of the power series on set $|z-z_0|\leq R_0$ as given by Theorem 5.63.2, there is $N_\varepsilon\in\mathbb{N}$ such that

$$|\rho_N(z)|<\frac{\varepsilon}{3} \text{ whenever } N>N_\varepsilon. \quad (**)$$

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Let $\varepsilon > 0$. If z is any point in some closed disk $|z-z_0| \leq R_0$ where $|z_1 - z_0| < R_0 < R_1$, then by the uniform convergence of the power series on set $|z-z_0| \leq R_0$ as given by Theorem 5.63.2, there is $N_\varepsilon \in \mathbb{N}$ such that

$$|\rho_N(z)| < \frac{\varepsilon}{3} \text{ whenever } N > N_\varepsilon. \quad (**)$$

Theorem 5.64.1 (continued 1)

Proof (continued). In particular, this inequality holds for each point z in some neighborhood $|z - z_1| < \delta_1$ of z_1 that is small enough to be contained in the disk $|z - z_1| \leq R_0$ (see Figure 81).

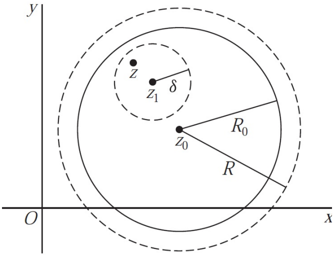

FIGURE 81

Now the partial sum $S_N(z)$ is a polynomial and so is continuous for each value of N at $z = z_1$ by Corollary 2.18.B. When $N = N_\varepsilon + 1$, by the definitions of continuity and limit, we can choose $\delta_2 > 0$ such that

$$|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3} \text{ whenever } |z - z_1| < \delta_2. \quad (***)$$

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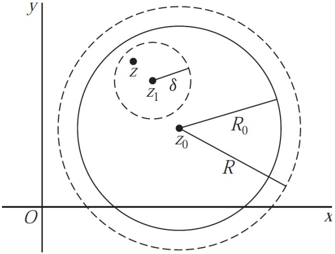

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Theorem 5.64.1 (continued 2)

Proof (continued). So with $N = N_\varepsilon + 1$, $\delta = \min\{\delta_1, \delta_2\}$, and with $|z - z_1| < \delta_2$ we have

$$\begin{aligned} |S(z) - S(z_1)| &\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \text{ by } (*) \\ &< \frac{\varepsilon}{3} + |\rho_N(z) + |\rho_N(z_1)| \text{ by } (***) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |\rho_N(z_1)| \text{ by } (**) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ with } z = z_1 \text{ in } (**) \\ &= \varepsilon. \end{aligned}$$

Therefore $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is continuous at z_1 and, since z_1 is an arbitrary point inside the circle of convergence, $S(z)$ is continuous inside the circle of convergence, as claimed. □

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$$\begin{array}{rcl} |S(z) - S(z_1)| & \leq & |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \text{ by } (*) \\ & < & \frac{\varepsilon}{3} + |\rho_N(z) + |\rho_N(z_1)| \text{ by } (***) \\ & < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |\rho_N(z_1)| \text{ by } (**) \\ & < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ with } z = z_1 \text{ in } (**) \\ & = & \varepsilon. \end{array}$$

Therefore $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is continuous at z_1 and, since z_1 is an arbitrary point inside the circle of convergence, $S(z)$ is continuous inside the circle of convergence, as claimed. □