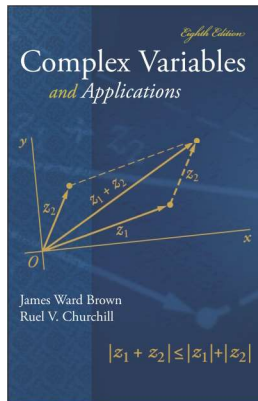


Complex Variables

Chapter 5. Series

Section 5.65. Integration and Differentiation of Power Series—Proofs of Theorems



Theorem 5.65.1

Theorem 5.65.1 (continued 1)

Proof (continued).

$$\begin{aligned} \int_C g(z)S(z) dz &= \int_C g(z)(S_N(z) + \rho_N(z)) dz \\ &= \int_C \left(g(z) \sum_{n=0}^{N-1} a_n(z-z_0)^n + g(z)\rho_N(z) \right) dz \\ &= \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n dz + \int_C g(z)\rho_N(z) dz. \quad (*) \end{aligned}$$

Now $|g(z)|$ has a maximum M on C by Theorem 2.18.3. Let L denote the length of C . Since power series $\rho_N(z)$ is uniformly convergent on C by Theorem 5.64.2 so for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|\rho_N(z)| < \varepsilon$ for all $N > N_\varepsilon$ and for all $z \in C$.

Theorem 5.65.1

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Theorem 5.65.1. Let C denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and let $g(z)$ be any function that is continuous on C . The series formed by multiplying each term of the power series by $g(z)$ can be integrated term-by-term over C ; that is

$$\int_C g(z)S(z) dz = \int_C g(z) \sum_{n=0}^{\infty} a_n(z-z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$$

Proof. Notice that $g(z)$ is continuous on C by hypothesis and $S(z)$ is continuous on C by Theorem 5.64.1, so $\int_C g(z)S(z) dz$ is defined. With $\rho_N(z)$ as the remainder $S(z) - S_N(z)$ (where $S_N(z)$ is the N th partial sum), by Note 4.40.A, ...

Theorem 5.65.1

Theorem 5.65.1 (continued 2)

Proof (continued). So (since N_ε is independent of $z \in C$) we have

$$\left| \int_C g(z)\rho_N(z) dz \right| < M\varepsilon L \text{ whenever } N > N_\varepsilon$$

by Theorem 4.43.A. Since M and L are constant (because C is given) then this last condition implies that $\lim_{N \rightarrow \infty} \int_C g(z)\rho_N(z) dz = 0$ by the definition of limit. Taking a limit as $N \rightarrow \infty$ for both sides of (*) we get

$$\begin{aligned} &\lim_{N \rightarrow \infty} \left(\int_C g(z)S(z) dz \right) \\ &= \lim_{N \rightarrow \infty} \left(\sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n dz + \int_C g(z)\rho_N(z) dz \right) \end{aligned}$$

or ...

Theorem 5.65.1 (continued 3)

Theorem 5.65.1. Let C denote any contour interior to the circle of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and let $g(z)$ be any function that is continuous on C . The series formed by multiplying each term of the power series by $g(z)$ can be integrated term-by-term over C ; that is

$$\int_C g(z)S(z) dz = \int_C g(z) \sum_{n=0}^{\infty} a_n(z - z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz.$$

Proof (continued). ...

$$\int_C g(z)S(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz,$$

as claimed. \square

Corollary 5.65.1

Corollary 5.65.1. The power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic at each point z interior to the circle of convergence of the series.

Proof. Let C be any closed contour in the domain which is the interior of the circle of convergence. With $g(z) = 1$ we then have

$$\int_C g(z)(z - z_0)^n dz = \int_C (z - z_0)^n dz = 0 \text{ for } n \in \mathbb{N} \cup \{0\}$$

by Theorem 4.44.A (or Example 4.43.A). So by Theorem 5.65.1,

$$\int_C g(z)S(z) dz = \int_C \sum_{n=0}^{\infty} a_n(z - z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz = 0.$$

Since C is an arbitrary closed contour in the circle of convergence of the series, then by Morera's Theorem (Theorem 4.52.2), $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic in the circle of convergence, as claimed. \square

Theorem 5.65.2

Theorem 5.65.2. The power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ can be differentiated term-by-term in its circle of convergence. That is, at each point z interior to the circle of convergence of that series, we have

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof. Let z be any point interior to the circle of convergence of the series, and let C be some positively oriented simple closed contour surrounding z and interior to the circle. Define $g(s) = \frac{1}{2\pi i} \frac{1}{(s - z)^2}$ for each $s \in C$. Since $z \notin C$ then g is continuous on C (as is $S(z)$), so by Theorem 5.65.1

$$\int_C g(s)S(s) ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n ds. \quad (*)$$

Theorem 5.65.2 (continued)

Proof (continued). Since $S(z)$ is analytic inside and on C by Corollary 5.65.1, then by Theorem 4.51.1 with $n = 1$ (and z_0 in the theorem as z here) we have

$$S'(z) = \frac{1}{2\pi i} \int_C \frac{S(s) ds}{(s - z)^2} = \int_C g(s)S(s) ds. \quad (**)$$

Similarly, replacing $S(z)$ with $(z - z_0)^n$ in $(**)$ we have

$$\frac{d}{dz} [(z - z_0)^n] = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n ds}{(s - z)^2} = \int_C g(s)(s - z_0)^n ds,$$

and so, combining $(*)$ and $(**)$, we have

$$\begin{aligned} \frac{d}{dz} \left[\sum_{n=0}^{\infty} a_n (z - z_0)^n \right] &= S'(z) = \int_C g(s)S(s) ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n ds \\ &= \sum_{n=0}^{\infty} a_n \frac{d}{dz} [(z - z_0)^n] = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \text{ as claimed. } \square \end{aligned}$$