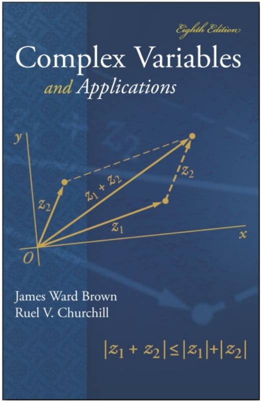
Complex Variables

Chapter 5. Series

Section 5.65. Integration and Differentiation of Power Series—Proofs of Theorems



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$$\int_C g(z)S(z)\,dz = \int_C g(z)\sum_{n=0}^{\infty} a_n(z - z_0)^n\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz.$$

Proof. Notice that $g(z)$ is continuous on C by hypothesis and $S(z)$ is continuous on C by Theorem 5.64.1, so $\int_C g(z)S(z)\,dz$ is defined. With $\rho_N(z)$ as the remainder $S(z) - S_N(z)$ (where $S_N(z)$ is the Nth partial sum), by Note 4.40.A, ...

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Theorem 5.65.1 (continued 1)

Proof (continued).

$$\begin{aligned}\int_C g(z)S(z)\,dz &= \int_C g(z)(S_N(z)+\rho_N(z))\,dz \\ &= \int_C \left(g(z)\sum_{n=0}^{N-1} a_n(z-z_0)^n + g(z)\rho_N(z)\right)\,dz \\ &= \sum_{n=1}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz. \qquad (*)\end{aligned}$$

Now $|g(z)|$ has a maximum M on C by Theorem 2.18.3. Let L denote the length of C. Since power series $\rho_N(z)$ is uniformly convergent on C by Theorem 5.64.2 so for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|\rho_N(z)| < \varepsilon$ for all $N > N_\varepsilon$ and for all $z \in C$.

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Theorem 5.65.1 (continued 2)

Proof (continued). So (since N_ε is independent of $z \in C$) we have

$$\left| \int_C g(z)\rho_N(z)\,dz \right| < M\varepsilon L \text{ whenver } N > N_\varepsilon$$

by Theorem 4.43.A. Since M and L are constant (because C is given) then this last condition implies that $\lim_{N\to\infty} \int_C g(z)\rho_N(z)\,dz = 0$ by the definition of limit. Taking a limit as $N \to \infty$ for both sides of $(*)$ we get

$$\lim_{N\to\infty} \left(\int_C g(z)S(z)\,dz \right)$$

$$= \lim_{N\to\infty} \left(\sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z)\rho_N(z)\,dz \right)$$

or ...

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$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz,$$

as claimed. □

Corollary 5.65.1

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Proof. Let C be any *closed* contour in the domain which is the interior of the circle of convergence. With $g(z) = 1$ we then have

$$\int_C g(z)(z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = 0 \text{ for } n \in \mathbb{N} \cup \{0\}$$

by Theorem 4.44.A (or Example 4.43.A).

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Since C is an arbitrary closed contour in the circle of convergence of the series, then by Morera's Theorem (Theorem 4.52.2), $S(z) = \sum_{n=0}^{\infty} (z - z_0)^n$ is analytic in the circle of convergence, as claimed. □

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Proof. Let z be any point interior to the circle of convergence of the series, and let C be some positively oriented simple closed contour surrounding z and interior to the circle. Define $g(s) = \frac{1}{2\pi i} \frac{1}{(s-z)^2}$ for each $s \in C$. Since $z \notin C$ then g is continuous on C (as is $S(z)$), so by Theorem 5.65.1

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Proof (continued). Since $S(z)$ is analytic inside and on C by Corollary 5.65.1, then by Theorem 4.51.1 with $n = 1$ (and z_0 in the theorem as z here) we have

$$S'(z) = \frac{1}{2\pi i}\int_C \frac{S(s)\,ds}{(s-z)^2} = \int_C g(s)S(s)\,ds. \qquad (**)$$

Similarly, replacing $S(z)$ with $(z-z_0)^n$ in $(**)$ we have

$$\frac{d}{dz}[(z-z_0)^n] = \frac{1}{2\pi i}\int_C \frac{(s-z_0)^n\,ds}{(s-z)^2} = \int_C g(s)(s-z_0)^n\,ds,$$

and so, combining $(*)$ and $(**)$, we have

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