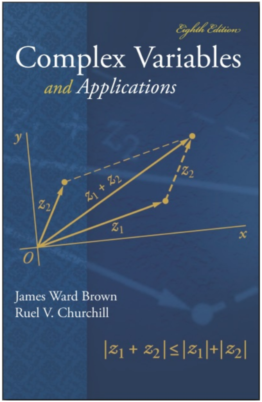
Complex Variables

Chapter 5. Series

Section 5.66. Uniqueness of Series Representations—Proofs of Theorems



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Theorem 5.66.1

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Proof. First, we reindex the series as $\sum_{m=0}^{\infty} a_m(z-z_0)^m$ where $|z-z_0| < R$. By Theorem 5.65.1,

$$\int_C g(z)f(z)\,dz = \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m\,dz \qquad (*)$$

where C is some circle centered at z_0 with radius less than R and $g(z)$ continuous on C; we consider

$$g(z) = \frac{1}{2\pi i}\frac{1}{(z-z_0)^{n+1}} \text{ where } n \in \mathbb{N} \cup \{0\}.$$

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Theorem 5.66.1 (continued)

Proof (continued). So

$$\begin{aligned}\int_C f(z)f(z)\,dz &= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \text{ by the definition of } g(z)\\ &= \frac{f^{(n)}(z_0)}{n!} \text{ by Theorem 4.51.1.} \qquad (**)\end{aligned}$$

By Exercise 4.42.10 (Exercise 4.46.13 in the 9th edition of the book)

$$\int_C g(z)(z-z_0)^m\,dz = \frac{1}{2\pi i}\int_C \frac{dz}{(z-z_0)^{n-m+1}} = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n, \end{cases}$$

so from $(*)$ and $(**)$ this implies

$$a_n = \sum +m = 0^\infty a_m \int_C g(z)(z-z_0)^m\,dz = \int_C g(z)f(z)\,dz = \frac{f^{(n)}(z_0)}{n!}.$$

But then by Taylor's Theorem (Theorem 5.57.A), this means that $\sum_{n=0}^\infty a_n(z-z_0)^n = \sum_{n=0}^\infty a_m(z-z_0)^m$ is the Taylor series for $f(z)$, as claimed. □

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Theorem 5.66.2

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$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z - z_0$ for that domain.

Proof. The proof is similar to the previous one. We reindex the series as $\sum_{m=-\infty}^{\infty} c_m(z-z_0)^m$. Let $g(z) = \dfrac{1}{2\pi i}\dfrac{1}{(z-z_0)^{n+1}}$ where $n \in \mathbb{Z}$ and let C be any circle around the annulus centered at z_0 and taken in the positive sense. By Theorem 5.65.1 and Exercise 5.66.10 (Exercise 5.72.10 in the 9th edition of the book) we have

$$\int_C g(z)f(z)\,dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-z_0)^m\,dz \ldots$$

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Theorem 5.66.2 (continued)

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$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-z_0)^m\,dz. \qquad (*)$$

By Exercise 4.42.10 (Exercise 4.46.13 in the 9th edition of the book) we have (even for m negative)

$$\int_C g(z)(z-z_0)^m\,dz = \frac{1}{2\pi i}\int_C \frac{dz}{(z-z_0)^{n-m+1}} = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n, \end{cases}$$

so from $(*)$ we have

$$c_n = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-z_0)^m\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}.$$

But then by Laurent's Theorem (Theorem 5.60.1), $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{m=-\infty}^{\infty} c_m(z-z_0)^m$ is the Laurent series of $f(z)$, as claimed. □

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