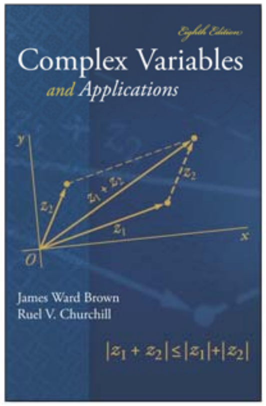
Complex Variables

Chapter 6. Residues and Poles

Section 6.70. Cauchy's Residue Theorem—Proofs of Theorems



Theorem 6.70.1

Theorem 6.70.1. Cauchy's Residue Theorem.
Let C be as simple closed contour described in the positive sense. If function f is analytic inside and on C except for a finite number of singular points z_k for $k = 1, 2, \ldots, n$ inside C then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z).$$

Proof. Since the points $z_1, z_2, \ldots, z_n$ are isolated then for each k with $k = 1, 2, \ldots, n$ there is $\varepsilon_k > 0$ such that the closed disc $|z - z_k| \le \varepsilon_k$ does not intersect C and does not intersect any of the other such closed discs around the points. We take the circle $|z - z + k| = \varepsilon_k$ and give it a positive orientation for $k = 1, 2, \ldots, n$ (see Figure 87).

Theorem 6.70.1 (continued)

Proof (continued).

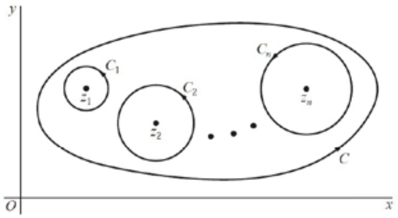


FIGURE 87

Then C along with $C_1, C_2, \ldots, C_n$ are the boundary of a closed region throughout which f is analytic. Notice that the region is multiply connected. So, by Theorem 4.49.A, $\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$ (notice that the C_k have a clockwise, i.e. negative, orientation in the statement of Theorem 4.49.A, so a negative sign is introduced here). By Note 69.A, $\int_{C_k} f(z)\,dz = 2\pi i \mathrm{Res}_{z=z_k} f(z)$ and so

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z), \text{ as claimed.} \qquad \square$$