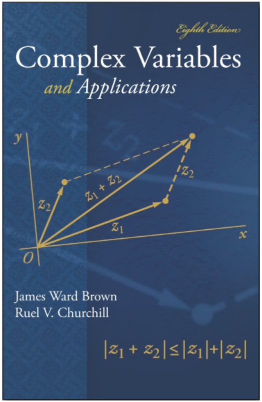
Complex Variables

Chapter 6. Residues and Poles

Section 6.71. Residues at Infinity—Proofs of Theorems



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Proof. Choose $R_0 > 0$ sufficiently large so that $C \subset \{z \mid |z| < R_0\}$ and define C_0 as the circle $|z| = R_0$ oriented in the negative direction. See Figure 89.

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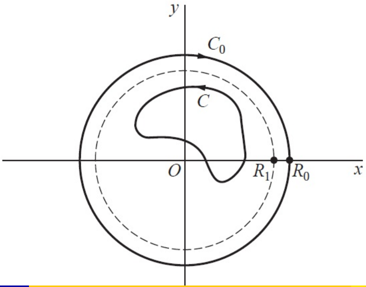

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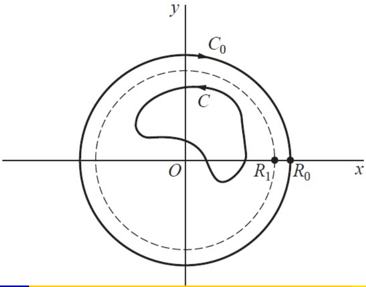

Theorem 6.71.1 (continued 1)

Proof (continued). By Theorem 4.49.B, "Principle of Deformation," $\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz$. Then, by the definition of residue at infinity, $\int_C f(z)\,dz = \text{Res}_{z=\infty} f(z)$. Now we take the Laurent series of f about $z_0 = 0$ (f may or may not be analytic at z_0) to get by Theorem 60.1, "Laurent's Theorem," $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for $R_1 < |z| < \infty$ where $R_1 < R_0$ is such that $C \subset \{z \mid |z| < R_1\}$ and $c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)\,dz}{z^{n+1}}$ for $n \in \mathbb{Z}$ (see the note after Theorem 60.1 for the concise expression of c_n).

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$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\sum_{n=-\infty}^{\infty} \frac{c_n}{z^n} = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} \text{ for } 0 < |z| < \frac{1}{R_1}.$$

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Theorem 6.71.1 (continued 2)

Proof (continued). With $n = 1$ we get

$$\text{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) = c_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz.$$

Now by the definition of $\text{Res}_{z=\infty} f(z)$,

$$\text{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz = \text{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right)$$

and

$$\int_C f(z)\,dz = 2\pi i \text{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right),$$

as claimed. □

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