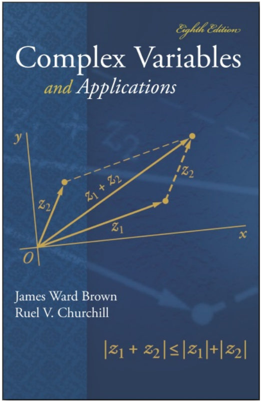
Complex Variables

Chapter 6. Residues and Poles

Section 6.73. Residues at Poles—Proofs of Theorems



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$$\text{Res}_{z=z_0} f(z) = \begin{cases} \varphi(z_0) & \text{if } m = 1 \\ \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} & \text{if } m \geq 2. \end{cases}$$

Proof. Suppose $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where φ is analytic and nonzero at z_0. Then by Theorem 5.57.A, "Taylor's Theorem,"

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z - z_0)^n \text{ for } |z| < R_2$$

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Theorem 6.73.1 (continued 1)

Proof (continued). ...

$$f(z) = \frac{1}{(z-z_0)^m}\varphi(z) = \frac{1}{(z-z_0)^m}\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= \sum_{n=0}^{\infty}\frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

$$= \frac{\varphi(z_0)}{(z-z_0)^m} + \frac{\varphi'(z_0)}{1!(z-z_0)^{m-1}} + \frac{\varphi''(z_0)}{2!(z-z_0)^{m-2}} + \cdots$$

$$+ \frac{\varphi^{(m-1)}(z_0)}{(m-1)!(z-z_0)} + \sum_{n=m}^{\infty}\frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}.$$

Since $\varphi(z_0) \neq 0$ then, by definition, f has a pole of order m at $z = z_0$ and $\text{Res}_{z=z_0} = \frac{\varphi^{(m-1)}(z_0}{(m-1)!}$ (notice this holds for $m = 1$ as well since $\varphi^{(0)}(z_0) = \varphi(z_0)$).

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Since $\varphi(z_0) \neq 0$ then, by definition, f has a pole of order m at $z = z_0$ and $\text{Res}_{z=z_0} = \frac{\varphi^{(m-1)}(z_0}{(m-1)!}$ (notice this holds for $m = 1$ as well since $\varphi^{(0)}(z_0) = \varphi(z_0)$).

Theorem 6.73.1 (continued 2)

Proof (continued). Now suppose f has a pole of order m at z_0. Then, by definition, the Laurent series of f centered at z_0 is of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}$$

for $0 < |z - z|0| < R_2$ for some $R_2 > 0$, where $b_m \neq 0$. Define $\varphi(z) = \begin{cases} (z - z_0)^m f(z) & \text{for } z \neq z_0 \\ b_m & \text{for } z = z_0. \end{cases}$ Then

$$\varphi(z) = b_m + b_{m-1}(z - z_0) + b_{m-2}(z - z_0)^2 + \cdots$$

$$+ b_1 (z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=-m}^{\infty} c_n (z - z_0)^n + m$$

for $0 < |z - z_0| < R_2$.

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$$\varphi(z) = b_m + b_{m-1}(z - z_0) + b_{m-2}(z - z_0)^2 + \cdots$$

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for $0 < |z - z_0| < R_2$.

Theorem 6.73.1 (continued 3)

Theorem 6.73.1. An isolated singular point z_0 of a function f has a pole of order m if and only if f can be written in the form $f(z) = \dfrac{\varphi(z)}{(z - z_0)^m}$ where φ is analytic for $|z| < R_2$ for some $R_2 > 0$, and $\varphi(z_0) \neq 0$.
Moreover,

$$\text{Res}_{z=z_0} f(z) = \begin{cases} \varphi(z_0) & \text{if } m = 1 \\ \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} & \text{if } m \geq 2. \end{cases}$$

Proof (continued). So φ is analytic in $|z - z_0| < R_2$ and $\varphi(z_0) = b_m \neq 0$, as claimed. In addition, by Theorem 5.57.A, "Taylor's Theorem,"

$$\text{Res}_{z=z_0} f(z) = b_1 = c_{-1} = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

□