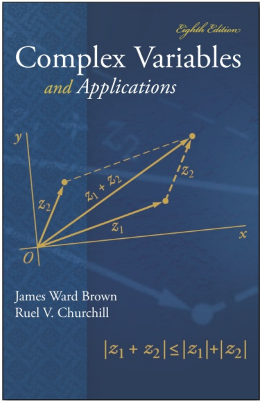
Complex Variables

Chapter 6. Residues and Poles

Section 6.75. Zeros of Analytic Functions—Proofs of Theorems



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Proof. First, by Taylor's Theorem (Theorem 5.57.!), f has a power series representation $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ valid throughout some neighborhood $|z - z_0| < \varepsilon$ of z_0.

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Suppose $f(z) = (z - z_0)^m g(z)$. Then since g is hypothesized to be analytic at z_0, then $g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n$ for some neighborhood $|z - z_0| < \varepsilon$ of z_0 (by Taylor's Theorem, Theorem 5.57.A). So $f(z) = (z - z_0)^m g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n+m}$ for $|z - z_0| < \varepsilon$.

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Proof (continued). So by Theorem 5.66.1 (Uniqueness of Power Series Representations)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n+m}$$

and so $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$. Also, $\frac{f^{(m)}(z_0)}{(m+1)!} = g(z_0) \neq 0$ by hypothesis and so $f^{(m)}(z_0) \neq 0$, as claimed.

Now suppose f has a zero of order m at z_0. That is, suppose $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

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$$f(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m}$$

$$= (z - z_0)^m \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n$$

for $|z - z_0| < \varepsilon$. So $f(z) = (z - z_0)^m g(z)$ where $g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n$ for $|z - z_0| < \varepsilon$. So g is analytic at z_0 and $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$, as claimed. $\square$

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Theorem 6.75.2

Theorem 6.75.2. Given a function f and a point z_0, suppose that

(a) f is analytic at z_0,

(b) $f(z_0) = 0$ but f is not identically equal to zero in any neighborhood of z_0.

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Proof. Since f is not identically equal to zero then not all derivatives of f at z_0 are 0 (or else, by Taylor's Theorem, Theorem 5.57.A, f has a series representation, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$, but if all derivatives of f at z_0 are 0 then the series of f is identically 0, a contradiction).

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Theorem 6.75.3

Theorem 6.75.3. Let f be a function and let z_0 a point where

(a) f is analytic throughout a neighborhood N_0 of z_0 and with power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $z \in N_0$, and

(b) $f(z) = 0$ at each point z of a domain D or a line segment L containing z_0.

Then $f(z) \equiv 0$ in N_0. That is, f is identically equal to zero throughout N_0.

Proof. Let f satisfy the stated conditions. ASSUME $f(z)$ is not identically zero in some neighborhood of z_0, but $f(z_0) = 0$. Then by Theorem 6.75.2, $f(z) \neq 0$ on some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. But this CONTRADICTS hypothesis (b).

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