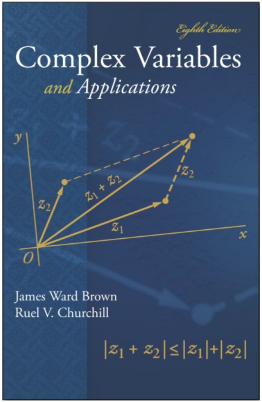
Complex Variables

Chapter 6. Residues and Poles

Section 6.76. Zeros and Poles—Proofs of Theorems



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Theorem 6.76.1

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(a) two functions p and q are analytic at a point z_0, and

(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Proof. Since q has a zero of order m at z_0 then by Theorem 6.75.2, there is a deleted neighborhood at z_0 throughout which $q(z) \neq 0$. So p/q has an isolated singular point at z_0 and by Theorem 6.75.1, we have $q(z) = (z - z_0)^m g(z)$ where g is analytic and nonzero at z_0.

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$$\frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_0)^m g(z)} = \frac{p(z)/g(z)}{(z - z_0)^m} = \frac{\varphi(z)}{(z - z_0)^m}$$

where $\varphi(z) = p(z)/q(z)$ is analytic and nonzero (by hypothesis (b)) are z_0. So by Theorem 6.73.1, z_0 is a pole of order m of p/q, as claimed. □

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Theorem 6.76.2

Theorem 6.76.2. Let the functions p and q be analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) = 0$ (that is, q has a zero of multiplicity one at z_0) then z_0 is a simple pole of p/q and $\text{Res}_{z=z_0} \dfrac{p(z)}{q(z)} = \dfrac{p(z_0)}{q'(z_0)}$.

Proof. By Theorem 6.75.1, $q(z) = (z - z_0)g(z)$ where g is analytic and nonzero at z_0. So by Theorem 6.76.1, p/q has a simple pole at z_0.

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$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)},$$

as claimed. □

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