Complex Variables

Chapter 6. Residues and Poles

Section 6.77. Behavior of Functions Near Isolated Singular Points—Proofs of Theorems

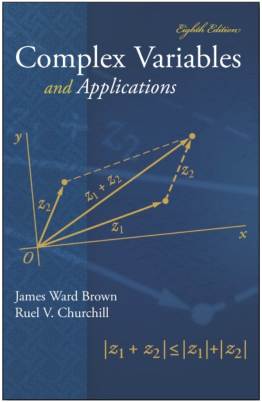

Table of contents

Theorem 6.77.1

Theorem 6.77.1. If z_0 is a pole of a function f then $\lim_{z \to z_0} f(z) = \infty$.

Proof. Suppose f has a pole of order m at $z = z_0$. Then by Theorem 6.73.1, $f(z) = \dfrac{\varphi(z)}{(z - z_0)^m}$ where φ is analytic for $|z - z_0| < R_2$ for some $R_2 > 0$ and $\varphi(z_0) \neq 0$.

Theorem 6.77.1

Theorem 6.77.1. If z_0 is a pole of a function f then $\lim_{z\to z_0} f(z) = \infty$.

Proof. Suppose f has a pole of order m at $z = z_0$. Then by Theorem 6.73.1, $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$ where φ is analytic for $|z - z_0| < R_2$ for some $R_2 > 0$ and $\varphi(z_0) \neq 0$. Then

$$\lim_{z\to z_0} \frac{1}{f(z)} = \lim_{z\to z_0} \frac{(z-z_0)^m}{\varphi(z)} = \frac{\lim_{z\to z_0}(z-z_0)^m}{\lim_{z\to z_0}\varphi(z)} = \frac{0}{\varphi(z_0} = 0.$$

So $\lim_{z\to z_0} f(z) = \infty$ by Theorem 2.17.1. □

Theorem 6.77.1

Theorem 6.77.1. If z_0 is a pole of a function f then $\lim_{z\to z_0} f(z) = \infty$.

Proof. Suppose f has a pole of order m at $z = z_0$. Then by Theorem 6.73.1, $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$ where φ is analytic for $|z - z_0| < R_2$ for some $R_2 > 0$ and $\varphi(z_0) \neq 0$. Then

$$\lim_{z\to z_0} \frac{1}{f(z)} = \lim_{z\to z_0} \frac{(z-z_0)^m}{\varphi(z)} = \frac{\lim_{z\to z_0}(z-z_0)^m}{\lim_{z\to z_0}\varphi(z)} = \frac{0}{\varphi(z_0} = 0.$$

So $\lim_{z\to z_0} f(z) = \infty$ by Theorem 2.17.1. □

Theorem 6.77.2

Theorem 6.77.2. If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Proof. Since a removable singular point is isolated (by definition) then f is analytic for $0 < |z - z_0| < R_2$ for some $R_2 > 0$. By Note 6.72.A, there is analytic g defined for $|z - z_0| < R_2$ such that $g(z) = f(z)$ for $0 < |z - z_0| < R_2$.

Theorem 6.77.2

Theorem 6.77.2. If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Proof. Since a removable singular point is isolated (by definition) then f is analytic for $0 < |z - z_0| < R_2$ for some $R_2 > 0$. By Note 6.72.A, there is analytic g defined for $|z - z_0| < R_2$ such that $g(z) = f(z)$ for $0 < |z - z_0| < R_2$. Let $\varepsilon > 0$ satisfy $\varepsilon < R_2$. Then g is continuous on $|z - z_0| \leq \varepsilon$ and so by Theorem 2.18.3 there is M such that $|g(z)| \leq M$ for all $|z - z_0| \leq \varepsilon$. Therefore $|f(z)| \leq M$ for all $0 < |z - z_0| \leq \varepsilon$ and the claim holds. □

Theorem 6.77.2

Theorem 6.77.2. If z_0 is a removable singular point of a function f, then f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

Proof. Since a removable singular point is isolated (by definition) then f is analytic for $0 < |z - z_0| < R_2$ for some $R_2 > 0$. By Note 6.72.A, there is analytic g defined for $|z - z_0| < R_2$ such that $g(z) = f(z)$ for $0 < |z - z_0| < R_2$. Let $\varepsilon > 0$ satisfy $\varepsilon < R_2$. Then g is continuous on $|z - z_0| \leq \varepsilon$ and so by Theorem 2.18.3 there is M such that $|g(z)| \leq M$ for all $|z - z_0| \leq \varepsilon$. Therefore $|f(z)| \leq M$ for all $0 < |z - z_0| \leq \varepsilon$ and the claim holds. □

Lemma 6.77.1. Riemann's Theorem

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof. Since f is analytic in $0 < |z - z_0| < \varepsilon$ then by Theorem 60.1, "Laurent's Theorem," there is a Laurent series for f centered at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \text{ for } 0 < |z-z_0| < \varepsilon.$$

Lemma 6.77.1. Riemann's Theorem

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof. Since f is analytic in $0 < |z - z_0| < \varepsilon$ then by Theorem 60.1, "Laurent's Theorem," there is a Laurent series for f centered at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \text{ for } 0 < |z-z_0| < \varepsilon.$$

Let C denote the positively oriented circle $|z - z_0| = \rho$ where $0 < \rho < \varepsilon$ (so that f is analytic on C). By Laurent's Theorem, $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$ for $n = 1, 2, \ldots$.

Lemma 6.77.1. Riemann's Theorem

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof. Since f is analytic in $0 < |z - z_0| < \varepsilon$ then by Theorem 60.1, "Laurent's Theorem," there is a Laurent series for f centered at z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \text{ for } 0 < |z-z_0| < \varepsilon.$$

Let C denote the positively oriented circle $|z - z_0| = \rho$ where $0 < \rho < \varepsilon$ (so that f is analytic on C). By Laurent's Theorem, $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$ for $n = 1, 2, \ldots$.

Lemma 6.77.1. Riemann's Theorem (continued)

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof (continued). Since f is hypothesized to be bounded on $0 < |z - z_0| < \varepsilon$, let M be the bound and then

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{-n+1}} \right| \\ &\leq \frac{1}{2\pi} \frac{M 2\pi \rho}{\rho^{-n+1}} \text{ by Theorem 4.43.A} \\ &= M\rho^n \text{ for } n = 1, 2, \ldots. \end{aligned}$$

Lemma 6.77.1. Riemann's Theorem (continued)

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof (continued). Since f is hypothesized to be bounded on $0 < |z - z_0| < \varepsilon$, let M be the bound and then

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \right| \\ &\leq \frac{1}{2\pi} \frac{M 2\pi\rho}{\rho^{-n+1}} \text{ by Theorem 4.43.A} \\ &= M\rho^n \text{ for } n = 1, 2, \ldots . \end{aligned}$$

Since $0 < \rho < \varepsilon$ is arbitrary, this inequality holds for all such ρ and hence $|b_n| = \lim_{\rho\to 0} |b_n| \leq \lim_{\rho \to 0} M\rho^n = 0$. That is, $b_n = c_n = 0$ for all $n - 1, 2, \ldots$ and the Laurent series for f is $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. So, by definition, the singular point z_0 of f is a removable singular point. □

Lemma 6.77.1. Riemann's Theorem (continued)

Lemma 7.77.1. Riemann's Theorem.
Suppose that a function f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0. If f is not analytic at z_0, then f has a removable singularity at z_0.

Proof (continued). Since f is hypothesized to be bounded on $0 < |z - z_0| < \varepsilon$, let M be the bound and then

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \right| \\ &\leq \frac{1}{2\pi} \frac{M 2\pi\rho}{\rho^{-n+1}} \text{ by Theorem 4.43.A} \\ &= M\rho^n \text{ for } n = 1, 2, \ldots . \end{aligned}$$

Since $0 < \rho < \varepsilon$ is arbitrary, this inequality holds for all such ρ and hence $|b_n| = \lim_{\rho\to 0} |b_n| \leq \lim_{\rho\to 0} M\rho^n = 0$. That is, $b_n = c_n = 0$ for all $n - 1, 2, \ldots$ and the Laurent series for f is $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. So, by definition, the singular point z_0 of f is a removable singular point. □

Theorem 6.73.1

Theorem 6.77.3. Casorati-Weierstrass Theorem
Suppose that z_0 is an essential singularity of function f and let w_0 be any complex number. Then for all $\varepsilon > 0$, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point a in every deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 for $\delta > 0$.

Proof. Let $w_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\delta > 0$ be given where δ is sufficiently small so that f is analytic on $0 < |z - z_0| < \delta$.

Theorem 6.73.1

Theorem 6.77.3. Casorati-Weierstrass Theorem
Suppose that z_0 is an essential singularity of function f and let w_0 be any complex number. Then for all $\varepsilon > 0$, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point a in every deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 for $\delta > 0$.

Proof. Let $w_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\delta > 0$ be given where δ is sufficiently small so that f is analytic on $0 < |z - z_0| < \delta$. ASSUME $|f(z) - w_0| \geq \varepsilon$ for all z in $0 < |z - z_0 < \delta$. Then the function $g(z) = 1/(f(z) - w_0)$ is analytic and bounded (by $M = 1/\varepsilon$) on $0 < |z - z_0| < \delta$ (notice that g is nonzero by the definition for these z values). So by Lemma 6.77.1, z_0 is a removable singularity of g.

Theorem 6.73.1

Theorem 6.77.3. Casorati-Weierstrass Theorem
Suppose that z_0 is an essential singularity of function f and let w_0 be any complex number. Then for all $\varepsilon > 0$, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point a in every deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 for $\delta > 0$.

Proof. Let $w_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\delta > 0$ be given where δ is sufficiently small so that f is analytic on $0 < |z - z_0| < \delta$. ASSUME $|f(z) - w_0| \geq \varepsilon$ for all z in $0 < |z - z_0 < \delta$. Then the function $g(z) = 1/(f(z) - w_0)$ is analytic and bounded (by $M = 1/\varepsilon$) on $0 < |z - z_0| < \delta$ (notice that g is nonzero by the definition for these z values). So by Lemma 6.77.1, z_0 is a removable singularity of g. We extend g to be defined at z_0 by setting $g(z_0) = \lim_{z \to z_0} g(z)$. Then g is analytic on $|z - z_0| < \delta$ (see Note 6.72.A).

Theorem 6.73.1

Theorem 6.77.3. Casorati-Weierstrass Theorem
Suppose that z_0 is an essential singularity of function f and let w_0 be any complex number. Then for all $\varepsilon > 0$, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point a in every deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 for $\delta > 0$.

Proof. Let $w_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\delta > 0$ be given where δ is sufficiently small so that f is analytic on $0 < |z - z_0| < \delta$. ASSUME $|f(z) - w_0| \geq \varepsilon$ for all z in $0 < |z - z_0 < \delta$. Then the function $g(z) = 1/(f(z) - w_0)$ is analytic and bounded (by $M = 1/\varepsilon$) on $0 < |z - z_0| < \delta$ (notice that g is nonzero by the definition for these z values). So by Lemma 6.77.1, z_0 is a removable singularity of g. We extend g to be defined at z_0 by setting $g(z_0) = \lim_{z \to z_0} g(z)$. Then g is analytic on $|z - z_0| < \delta$ (see Note 6.72.A).

Theorem 6.73.1 (continued)

Proof (continued). If $g(z_0) \neq 0$ then $f(z) = \frac{1}{g(z)} + w_0$ and f is analytic where g is nonzero. Since $g(z_0) \neq 0$ then $g(z) \neq 0$ for $|z - z_0| < \delta$. But then f is analytic on $0 < |z - z_0| < \delta$ and $\lim_{z \to z_0} f(z) = \frac{1}{g(z_0)} + w_0$. So from the definition of limit, there is δ_1 such that $0 < \delta_1 < \delta$ and f is bounded on $0 < |z - z_0| < \delta_1$. But then, by Lemma 6.77.1, f has a removable singular point at $z = z_0$, not an essential singularity, a CONTRADICTION.

Theorem 6.73.1 (continued)

Proof (continued). If $g(z_0) \neq 0$ then $f(z) = \frac{1}{g(z)} + w_0$ and f is analytic where g is nonzero. Since $g(z_0) \neq 0$ then $g(z) \neq 0$ for $|z - z_0| < \delta$. But then f is analytic on $0 < |z - z_0| < \delta$ and $\lim_{z \to z_0} f(z) = \frac{1}{g(z_0)} + w_0$. So from the definition of limit, there is δ_1 such that $0 < \delta_1 < \delta$ and f is bounded on $0 < |z - z_0| < \delta_1$. But then, by Lemma 6.77.1, f has a removable singular point at $z = z_0$, not an essential singularity, a CONTRADICTION.

If $g(z_0) = 0$ then, since g is not identically the zero function (since g is nonzero for $0 < |z - z_0| < \delta$) then z_0 is a zero of g of some order m (see Section 75) and so by Theorem 6.76.1 (with $p(z) = 1 + g(z)w_0$ and $q(z) = g(z)$), $f(z) = \frac{1}{g(z)} + w_0 = \frac{1+g(z)w_0}{g(z)}$ has a pole of order m at z_0, CONTRADICTING the fact that f has an essential singularity, no a pole at z_0.

Theorem 6.73.1 (continued)

Proof (continued). If $g(z_0) \neq 0$ then $f(z) = \frac{1}{g(z)} + w_0$ and f is analytic where g is nonzero. Since $g(z_0) \neq 0$ then $g(z) \neq 0$ for $|z - z_0| < \delta$. But then f is analytic on $0 < |z - z_0| < \delta$ and $\lim_{z \to z_0} f(z) = \frac{1}{g(z_0)} + w_0$. So from the definition of limit, there is δ_1 such that $0 < \delta_1 < \delta$ and f is bounded on $0 < |z - z_0| < \delta_1$. But then, by Lemma 6.77.1, f has a removable singular point at $z = z_0$, not an essential singularity, a CONTRADICTION.

If $g(z_0) = 0$ then, since g is not identically the zero function (since g is nonzero for $0 < |z - z_0| < \delta$) then z_0 is a zero of g of some order m (see Section 75) and so by Theorem 6.76.1 (with $p(z) = 1 + g(z)w_0$ and $q(z) = g(z)$), $f(z) = \frac{1}{g(z)} + w_0 = \frac{1+g(z)w_0}{g(z)}$ has a pole of order m at z_0, CONTRADICTING the fact that f has an essential singularity, no a pole at z_0. So the assumption that $|f(z) - w_0| \geq \varepsilon$ for all $0|z - z_0| < \delta$ is false and so there must be some point z in $0 < |z - z_0| < \delta$ such that all $0 < |z - z_0| < \delta$ is false and so there must be some point z in $0 < |z - z_0| < \delta$ such that $|f(z) - w_0| < \varepsilon$, as claimed. □

Theorem 6.73.1 (continued)

Proof (continued). If $g(z_0) \neq 0$ then $f(z) = \frac{1}{g(z)} + w_0$ and f is analytic where g is nonzero. Since $g(z_0) \neq 0$ then $g(z) \neq 0$ for $|z - z_0| < \delta$. But then f is analytic on $0 < |z - z_0| < \delta$ and $\lim_{z \to z_0} f(z) = \frac{1}{g(z_0)} + w_0$. So from the definition of limit, there is δ_1 such that $0 < \delta_1 < \delta$ and f is bounded on $0 < |z - z_0| < \delta_1$. But then, by Lemma 6.77.1, f has a removable singular point at $z = z_0$, not an essential singularity, a CONTRADICTION.

If $g(z_0) = 0$ then, since g is not identically the zero function (since g is nonzero for $0 < |z - z_0| < \delta$) then z_0 is a zero of g of some order m (see Section 75) and so by Theorem 6.76.1 (with $p(z) = 1 + g(z)w_0$ and $q(z) = g(z)$), $f(z) = \frac{1}{g(z)} + w_0 = \frac{1+g(z)w_0}{g(z)}$ has a pole of order m at z_0, CONTRADICTING the fact that f has an essential singularity, no a pole at z_0. So the assumption that $|f(z) - w_0| \geq \varepsilon$ for all $0|z - z_0| < \delta$ is false and so there must be some point z in $0 < |z - z_0| < \delta$ such that all $0 < |z - z_0| < \delta$ is false and so there must be some point z in $0 < |z - z_0| < \delta$ such that $|f(z) - w_0| < \varepsilon$, as claimed. □