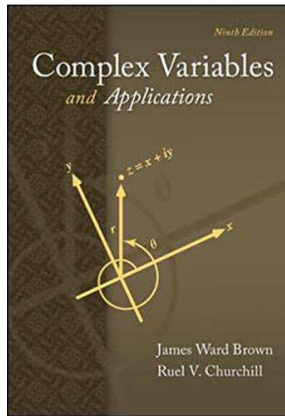


# Complex Variables

## Chapter 9. Conformal Mapping

### Section 115. Harmonic Conjugates—Proofs of Theorems



## Theorem 115.B

**Theorem 115.B.** If a harmonic function  $u(x, y)$  is defined on a simply connected domain  $D$ , it always has a harmonic conjugate  $v(x, y)$  in  $D$ .

**Proof.** We need another result from Advanced Calculus. We quote from Brown and Churchill. "Suppose that  $P(x, y)$  and  $Q(x, y)$  have continuous first-order partial derivatives in a simply connected domain  $D$  of the  $xy$ -plane, and let  $(x_0, y_0)$  and  $(x, y)$  be any two points in  $D$ . If  $P_y = Q_x$  everywhere in  $D$ , then the line integral  $\int_C (P(s, t) ds + Q(s, t) dt)$  from  $(x_0, y_0)$  to  $(x, y)$  is independent of the contour  $C$  that is taken as long as the contour lies entirely in  $D$ . Furthermore, when the point  $(x_0, y_0)$  is kept fixed and  $(x, y)$  is allowed to vary throughout  $D$ , the integral represents a single-valued function

$$F(x, y) = \int_{(x_0, y_0)}^{(x, y)} (P(s, t) ds + Q(s, t) dt)$$

of  $x$  and  $y$  whose first-order partial derivatives are given . . .

## Theorem 115.B (continued 1)

**Theorem 115.B.** If a harmonic function  $u(x, y)$  is defined on a simply connected domain  $D$ , it always has a harmonic conjugate  $v(x, y)$  in  $D$ .

**Proof (continued).** . . . by the equations  $F_x(x, y) = P(x, y)$ ,  $F_y(x, y) = Q(x, y)$ ." If a different initial point other than  $(x_0, y_0)$  is chosen, then this will change  $F$  by an additive constant (the "constant of integration").

Since  $u$  is hypothesized to be harmonic, then  $u_{xx} + u_{yy} = 0$ , or  $(-u_y)_y = (u_x)_x$  in  $D$ . The second partials of  $u$  are continuous in  $D$  (by the definition of "harmonic function"), so the first-order partial derivatives of  $u$  are continuous. With  $(x_0, y_0)$  as a fixed point in  $D$ , by the result above (with  $P(s, t) = -u_t(s, t)$  and  $Q(s, t) = u_s(s, t)$ ) we have that the function

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} (-u_t(s, t) ds + u_s(s, t) dt)$$

is well defined for all  $(x, y)$  in  $D$ .

## Theorem 115.B (continued 2)

**Theorem 115.B.** If a harmonic function  $u(x, y)$  is defined on a simply connected domain  $D$ , it always has a harmonic conjugate  $v(x, y)$  in  $D$ .

**Proof (continued).** With  $F(x, y) = v(x, y)$ , we also have by the above result that  $v_x(x, y) = -u_y(x, y)$  and  $v_y(x, y) = u_x(x, y)$ . So this means  $u$  and  $v$  satisfy the Cauchy-Riemann equations. Since the first-order partials of  $u$  are continuous, then the Cauchy-Riemann equations imply that the first-order partial derivatives of  $v$  are also continuous. Now by Theorem 2.22.A, "The Cauchy-Riemann Equations and Continuity Imply Differentiability,"  $u(x, y) + iv(x, y)$  is an analytic function in  $D$ , and so  $v$  is a harmonic conjugate of  $u$ , as claimed. We can also add and constant  $C$  to  $v(x, y)$ , to get  $v(x, y) + C$  as another harmonic conjugate of  $u$  (this just corresponds to a different choice of point  $(x_0, y_0)$  in the integral above). □