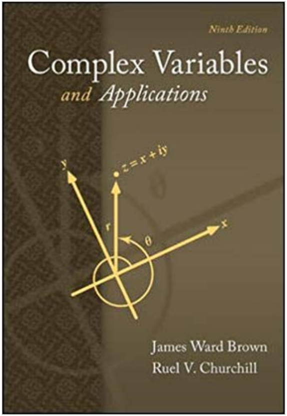
Complex Variables

Chapter 9. Conformal Mapping

Section 115. Harmonic Conjugates—Proofs of Theorems



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Theorem 115.B

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Proof. We need another result from Advanced Calculus. We quote from Brown and Churchill. "Suppose that $P(x, y)$ and $Q(x, y)$ have continuous first-order partial derivatives in a simply connected domain D of the xy-plane, and let (x_0, y_0) and (x, y) be any two points in D. If $P_y = Q_x$ everywhere in D, then the line integral $\int_C (P(s, t)\, ds + Q(s, t)\, dt)$ from (x_0, y_0) t (x, y) is independent of the contour C that is taken as long as the contour lies entirely in D.

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$$F(x,y) = \int_{(x_0,y_0)}^{(x,y)} (P(s,t)\,ds + Q(s,t)\,dt)$$

of x and y whose first-order partial derivatives are given ...

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Theorem 115.B (continued 1)

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Proof (continued). ...by the equations $F_x(x, y) = P(x, y)$, $F_y(x, y) = Q(x, y)$." If a different initial point other than (x_0, y_0) is chosen, then this will change F by an additive constant (the "constant of integration).

Since u is hypothesized to be harmonic, then $u_{xx} + u_{yy} = 0$, or $(-u_y)_y = (u_x)_x$ in D. The second partials of u are continuous in D (by the definition of "harmonic function"), so the first-order partial derivatives of u are continuous. With (x_0, y_0) as a fixed point in D, by the result above (with $P(s, t) = -u_t(s, t)$ and $Q(s, t) = u_s(s, t)$) we have that the function

$$v(x, y) = \int_{(x_0,y_0)}^{(x,y)} (-u_t(s, t)\, ds + u_s(s, t)\, dt)$$

is well defined for all (x, y) in D.

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Theorem 115.B (continued 2)

Theorem 115.B. If a harmonic function $u(x, y)$ is defined on a simply connected domain D, it always has a harmonic conjugate $v(x, y)$ in D.

Proof (continued). With $F(x, y) = v(x, y)$, we also have by the above result that $v_x(x, y) = -u_y(x, y)$ and $v_y(x, y) = u_x(x, y)$. So this means u and v satisfy the Cauchy-Riemann equations. Since the first-order partials of u are continuous, then the Cauchy-Riemann equations imply that the first-order partial derivatives of v are also continuous. Now by Theorem 2.22.A, "The Cauchy-Riemann Equations and Continuity Imply Differentiability," $u(x, y) + iv(x, y)$ is an analytic function in D, and so v is a harmonic conjugate of u, as claimed. We can also add and constant C to $v(x, y)$, to get $v(x, y) + C$ as another harmonic conjugate of u (this just corresponds to a different choice of point (x_0, y_0) in the integral above). □

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