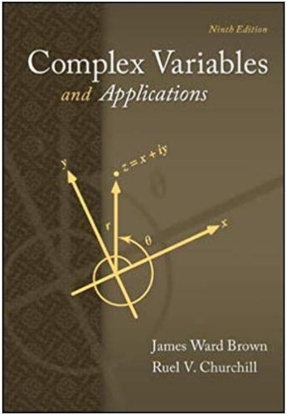
Complex Variables

Chapter 9. Conformal Mapping

Section 116. Transformations of Harmonic Functions—Proofs of Theorems



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Theorem 116.A

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(a) an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_z in the z plane onto a domain D_w in the w plane;

(b) $h(u, v)$ is a harmonic function defined on D_w.

It follows that the function $H(x, y) = h[(u(x, y), v(x, y)]$ is harmonic in D_z.

Proof. Recall that a domain is, by definition, a nonempty connected open set. First, suppose domain D_w is simply connected. By Theorem 115.B, harmonic function $h(u, v)$ has a harmonic conjugate $g(u, v)$. So by Theorem 2.26.1/115.A the function $\Phi(w) = h(u, v) + ig(u, v)$ is analytic in D_w. Since function f is analytic in D_z, the composite function $\Phi(f(z))$ is also analytic in D_z. Also by Theorem 2.26.1/115.A, the real part of $\Phi(f(z))$, $h(u(x, y), v(x, y))$, is harmonic in D_z, as claimed.

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Theorem 116.A (continued)

Proof (continued). Second, suppose domain D_w is not simply connected. Since D_w is open, then by Exercise 1.12.6, every point is an interior point so that for any point $w_0 \in D_w$ there is $\varepsilon > 0$ such that the neighborhood of w_0, $N_{w_0} = \{w \in \mathbb{C} \mid |w - w_0| < \varepsilon\}$, lies entirely in D_w. Since that neighborhood N_{w_0} of w_0 is simply connected, there is a function of the type $\Phi(w) = h(u, v) + ig(u, v)$ analytic in N_{w_0}. Since f is continuous at $z_0 \in D_z$ and $f(z_0) = w_0$, then by the definition of continuity, there is a neighborhood of z_0, $N_{z_0} = \{z \in \mathbb{C} \mid |z - z_0| < \delta\}$, whose image is contained in $N_{w_0} = \{w \in \mathbb{C} \mid |w - w_0| < \varepsilon\}$. So by the first part of the proof (with D_w there replaced with N_{w_0} here), we have that $h[(x, y), v(x, y)]$ is analytic on N_{z_0}. Since w_0 is an arbitrary point of D_w and since D_z is mapped onto D_w by $w = f(z)$ by hypothesis, then the function $h(u(x, y), v(x, y))$ is harmonic through D_z. □

Theorem 116.A (continued)

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