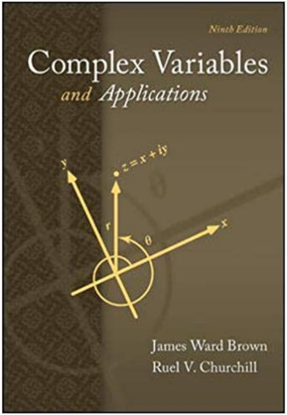
Complex Variables

Chapter 9. Conformal Mapping

Section 117. Transformations of Boundary Conditions—Proofs of Theorems



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Theorem 117.A

Theorem 117.A. Suppose that

(a) a transformation $w = f(z) = u(x, y) + iv(x, y)$ is conformal at each point of a smooth arc C and that Γ is the image of C under that transformation;

(b) $h(u, v)$ is a function that satisfies one of the conditions $h = h_0$ and $dh/dn = 0$ at points on Γ, where h_0 is a real constant and dh/dn denotes the directional derivatives of h normal to Γ.

It follows that the function $H(x, y) = h[(u(x, y), v(x, y)]$ satisfies the corresponding condition $H = h_0$ or $dH/dN = 0$ at points on C, where dH/dN denotes directional derivatives of H normal to C.

Proof. First, suppose that $h = h_0$ on Γ. Since $H(x, y) = h[u(x, y), v(x, y)]$ and $(x, y) \in C$ implies $(u(x, y), v(x, y)) \in \Gamma$, then $h = h(u, v) = h_0$ on Γ implies $H(x, y) = h[u(x, y), v(x, y)] = h_0$ for $(x, y) \in C$, as claimed.

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Theorem 117.A (continued 1)

Proof (continued). Second, suppose $dh/dn = 0$ on Γ. With $\mathbf{n}$ as a two dimensional unit vector normal to Γ at (u, v), we have by Note 117.A that the directional derivative of h at point (u, v) is $dh/dn = (\nabla h) \cdot \mathbf{n}$. Since by hypothesis $dh/dn = 0$ at (u, v), then we see that either $\nabla h = \mathbf{0}$ (which we deal with in Note 117.B) or ∇h is orthogonal to $\mathbf{n}$ at (u, v) (so that the dot product is 0). Since $\mathbf{n}$ is normal to Γ, then $\mathbf{n}$ is tangent to Γ (see Figure 151, right).

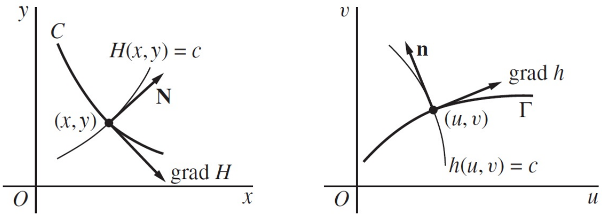

FIGURE 151

Theorem 117.A (continued 1)

Proof (continued). But gradients of f are orthogonal to level curves of f (defined by $f(x, y) = k$ for some constant k) by Theorem 14.5.B in my online notes for Calculus 3 (MATH 2110) on Section 14.5. Directional Derivatives and Gradient Vectors. Since ∇h is tangent to Γ and orthogonal to level curves, then the level curve $h(u, v) = c$ passing through (u, v) must be orthogonal to Γ. The level curve $H(x, y) = c$ in the z plane can be written as $H(x, y) = h[u(x, y), v(x, y)] = c$. Now C is transformed onto Γ by $w = f(z)$ by hypothesis, and Γ is orthogonal to level curve $h(u, v) = c$ (as shown above), so by the (hypothesized) conformality of f we have that C is orthogonal to the level curve $H(x, y) = c$ at the point (x, y) corresponding to point (u, v).

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