Chapter 1. Complex Numbers

Note. This chapter explores some algebraic and geometric properties of complex numbers. Functions and their derivatives are introduced in Chapter 2, more functions and their inverses are given in Chapter 3, integrals are studied in Chapter 4, and Series are addressed in Chapter 5. This complex variables class will cover these five chapters and possibly additional material from later in the book.

Section 1. Sums and Products

Note 1.1.A. There are a number of ways to define the field of complex numbers, $\mathbb{C}$. One way is to define it as an extension field of the real numbers, $\mathbb{C} = \mathbb{R}[i]$ where i is a root of the real polynomial $x^2 + 1$ (see the last page of my online notes from Introduction to Modern Algebra [MATH 4127/5127] on VI.29. Introduction to Extension Fields).

Definition. The field of *complex numbers*, $\mathbb{C}$, is the set of ordered pairs of real numbers: $\mathbb{C} = \{(x, y) \mid x, y \in \mathbb{R}\}$, where addition is defined as

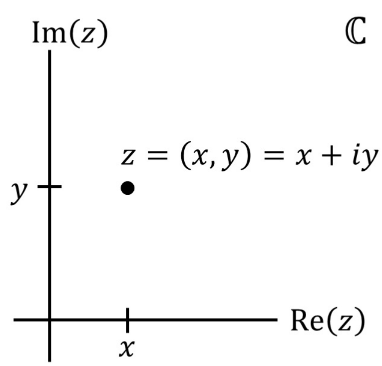
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and multiplication is defined as

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2).$$

Note. The previous definition is (in part) the "Cayley-Dickson Construction" which allows us to make a 2-dimensional *algebra* $\mathbb{C}$ based on $\mathbb{R}$. This process can be iterated to produce number systems beyond the complex number, namely the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$, and others. For more on this, see my online notes for Complex Analysis 1 (MATH 5510) on Section I.2. The Field of Complex Numbers; notice Notes 1.2.A and 1.2.B. More details and some history of this technique are given in my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Supplement. The Cayley-Dickson Construction and Nonassociative Algebras, and in the ETSU Abstract Algebra Club PowerPoint presentation on Numbers: The Reals and Beyond!.

Note. As you might expect, we denote $z = (x, y) \in \mathbb{C}$ as $z = x+iy$. Geometrically, $\mathbb{C}$ is the same as the Cartesian plane, $\mathbb{R}^2$ (however they are different algebraically; we do not multiply elements of $\mathbb{R}^2$ together, say... though if we take $\mathbb{R}^2$ as a vector space, then we can add elements). So we can associate any element of $\mathbb{C}$ with a point in $\mathbb{R}^2$. When doing so, we call the x-axis the "real axis" and call the y-axis the "imaginary axis." We have:



For $z \in \mathbb{C}$, we denote $x = \text{Re}(z)$ (the "real part of z") and $y = \text{Im}(z)$ (the "imaginary part of z"). An older notation is $x = \Re(z)$ and $y = \Im(z)$. Notice that $z = \text{Re}(z) + i\text{Im}(z)$, where $i = (0, 1)$. The figure above is an example of an "Argand diagram," named after Jean Robert Argand (July 18, 1768–August 13, 1822). However, this idea was first published by Caspar Wessel (June 8, 1745–March 25, 1818) in 1799. For more on this history, see Notes 1.3.A and 1.3.B in my online Complex Analysis 1 notes on Section I.3. The Complex Plane.

Note. With $(0, 1)$ denoted as i, we have by the definition of multiplication that

$$i^2 = (0,1)(0,1) = ((0)(0) - (1)(1), (1)(0) + (0)(1)) = (-1, 0) = -1 + i0 = -1.$$

Note. By writing $(x, y) = x + iy$, the definition of addition in $\mathbb{C}$ becomes

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

and the definition of multiplication becomes

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

$$= (x_1)(x_2) + (x_1)(iy_2) + (iy_1)(x_2) + (iy_1)(iy_2) \text{ ("FOIL")}.$$

So addition and multiplication in $\mathbb{C}$ satisfy the usual (field) properties. We now abandon the ordered pair notation (though we will return to the geometric presentation of $\mathbb{C}$ in terms of $\mathbb{R}^2$) and use the notation $z = x + iy$ introduced above. When we *do* appeal to the geometric plane representation, we call this the "complex plane" (and use this term synonymously with the terms "complex field" or "complex numbers").