Section 1.2. Basic Algebraic Properties

Note. We use several algebraic properties of the real numbers to verify corresponding properties of complex numbers. As we should in a course at this level, we give the results in a theorem/proof format.

Theorem 1.2.1. For any $z_1, z_2, z_3 \in \mathbb{C}$ we have the following.

1. Commutivity of addition and multiplication:

$$z_1 + z_2 = z_2 + z_1 \text{ and } z_1 z_2 = z_2 z_1.$$

2. Associativity of addition and multiplication:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ and } (z_1 z_2) z_3 = z_1(z_2 z_3).$$

3. Distribution of multiplication over addition:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

4. There is an *additive identity* $0 = 0 + i0$ such that $0 + z = z$ for all $z \in \mathbb{C}$. There is a *multiplicative identity* $1 = 1 + i0$ such that $z1 = z$ for all $z \in \mathbb{C}$. Also, $z0 = 0$ for all $z \in \mathbb{C}$.

5. For each $z \in \mathbb{C}$ there is $z' \in \mathbb{C}$ such that $z' + z = 0$. z' is the *additive inverse* of z (denoted $-z$). If $z \neq 0$, then there is $z'' \in \mathbb{C}$ such that $z''z = 1$. z'' is the *multiplicative inverse* of z (denoted z^{-1}).

Note. In Exercise 1.2.8, you will show that the additive and multiplicative identities are unique (so are the additive and multiplicative inverses of a given $z \in \mathbb{C}$).

Corollary 1.2.2. For $z_1, z_2, z_3 \in \mathbb{C}$ if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$. That is, $\mathbb{C}$ has no "zero divisors."

Note 1.2.A. When we consider division it is, by definition, multiplication by the multiplicative inverse. So for $z \neq 0$, z^{-1} is denoted $1/z$ and so z_1/z_2 means $z_1 z_2^{-1}$. We see in the proof of Theorem 1.2.1(5) that for $z = x + iy$,

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

So by the definition of multiplication in $\mathbb{C}$, for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ we have

$$z_1 z_2^{-1} = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

When we introduce moduli in Section 1.4 and complex conjugates in Section 1.5 we will have more streamlined and easy-to-remember formulas.

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