Section 1.4. Vectors and Moduli

Note. In this section, we associate a vector in $\mathbb{R}^2$ with a complex number. The length of the corresponding vector will be the "modulus" of the complex number. We should comment that this does not mean that $\mathbb{R}^2$ and $\mathbb{C}$ are the same vector space. $\mathbb{R}^2$ is a two dimensional vector space with real scalars, whereas $\mathbb{C}$ is a one dimensional vector space with complex scalars. Recall that the definition of a vector space isomorphism requires the relevant vector spaces to have the same scalar field. See my online notes Linear Algebra (MATH 2010) on 3.3. Coordinatization of Vectors. We prefer to say that the association of $\mathbb{C}$ with $\mathbb{R}^2$ gives a "geometric representation" of $\mathbb{C}$ and we refer to $\mathbb{R}^2$ in this association as the *complex plane*.

Definition. For $z = x + iy \in \mathbb{C}$, associate the vector $\langle x, y\rangle \in \mathbb{R}^2$. The *modulus* (or "absolute value") of z is

$$|z| = \|\langle x, y\rangle\| = \sqrt{x^2 + y^2} = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}.$$

(Notice that $|z| = | - z|$.)

Note. Addition of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ corresponds to the addition of the corresponding vectors $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \text{ and } \langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$

This gives us a parallelogram property of addition of complex numbers, similar to that of vectors. Notice that the modulus of z, $|z|$, corresponds to the distance of complex number z from the origin of the complex plane. Similarly, the distance between z_1 and z_2 is $|z_1 - z_2|$.

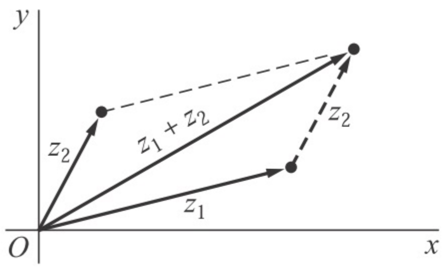

FIGURE 3

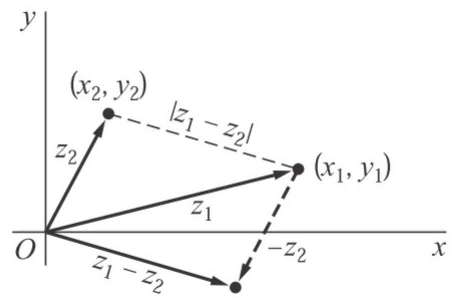

FIGURE 4

Note. The complex numbers $z \in \mathbb{C}$ satisfying the relationship $|z - z_0| = R$ for given $z_0 \in \mathbb{C}$ and $R \in \mathbb{R}$, $R > 0$, lie on circle with center z_0 and radius R.

Theorem. The Triangle Inequality.
For all $z_1, z_2 \in \mathbb{C}$, we have $|z_1 + z_2| \leq |z_1| + |z_2|$.

Note. Figure 3 above gives a geometric inspiration that makes the Triangle Inequality plausible. It also gives motivation for the name of the inequality, since $|z_1 + z_2|$, $|z_1|$, and $|z_2|$ are the lengths of the sides of a triangle. An algebraic proof is outlined in Exercise 1.5.15 (in Exercise 1.6.15 in the 9th edition of the book).

Corollary 1.4.1. For all $z_1, z_2 \in \mathbb{C}$, we have

$$||z_1| - |z_2|| \leq |z_1 + z_2|.$$

Note. Other easily established inequalities are:

- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$.
- $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots |z_n|$ for all $z_1, z_2, \ldots, z_n \in \mathbb{C}$.

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