Section 1.5. Complex Conjugates

Note. In this section, we introduce a useful operation on a complex number with an easy geometric interpretation.

Definition. The *complex conjugate* (or simply *conjugate*) of $z = x+iy$ is $\overline{z} = x-iy$.

Note. In the complex plane, $\overline{z}$ is the mirror image of z about the real axis (the "x-axis").

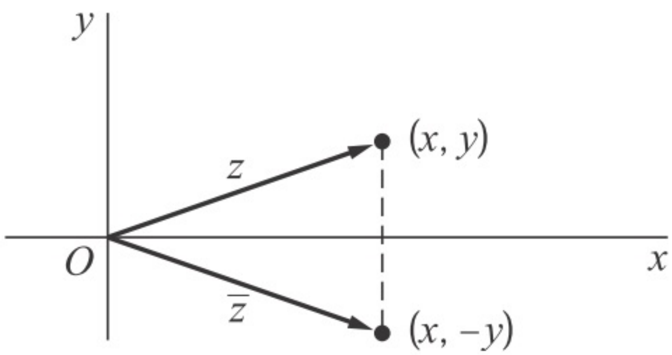

FIGURE 5

Theorem 1.5.1. For all $z_1, z_2 \in \mathbb{C}$ we have

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2 \qquad \overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$$
$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \qquad \overline{z_1 / z_2} = \overline{z}_1 / \overline{z}_2$$
$$\mathrm{Re}(z) = (z + \overline{z})/2 \qquad \mathrm{Im}(z) = (z - \overline{z})/(2i)$$

Note 1.5.A. Since with $z = x + iy$ we have $|z|^2 = |x + iy|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\overline{z}$, then the conjugate gives us a concise way to express the modulus and reciprocal of $z \in \mathbb{C}$:

$$|z|^2 = z\overline{z} \text{ and } \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

Theorem 1.5.2. For all $z_1, z_2 \in \mathbb{C}$ we have

$$|z_1 z_2| = |z_1||z_2| \text{ and } \left|\frac{z_1}{z_2}\right| = |z_1|/|z_2| \text{ (for } z \neq 0).$$

Revised: 1/22/2024