Section 1.6. Exponential Form

Note. Recall that a point $(x, y) \in \mathbb{R}^2$ can be represented by polar coordinates (r, θ) where $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ for $x \neq 0$; $x = r\cos\theta$ and $y = r\sin\theta$. Similarly, we represent $z \in \mathbb{C}$, $z \neq 0$, in "polar form" $z = r(\cos\theta + i\sin\theta)$. It is traditional in complex analysis to take $r = |z| > 0$ (as opposed to the $\mathbb{R}^2$ setting where r may be 0 or negative; this is the convention in Calculus 3 [MATH 2110], as seen in my online notes for that class on Section 11.3. Polar Coordinates).

Definition. For $z \in \mathbb{C}$, $z \neq 0$, if $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$, then any θ satisfying the equation is an *argument* of z. The set of all arguments z is denoted $\arg(z) = \{\theta \in \mathbb{R} \mid z = r(\cos\theta + i\sin\theta)\}$. The *principal value of arg*(z) is the unique $\theta \in \arg(z)$ such that $-\pi < \theta \leq \pi$ and is denoted $\mathrm{Arg}(z)$.

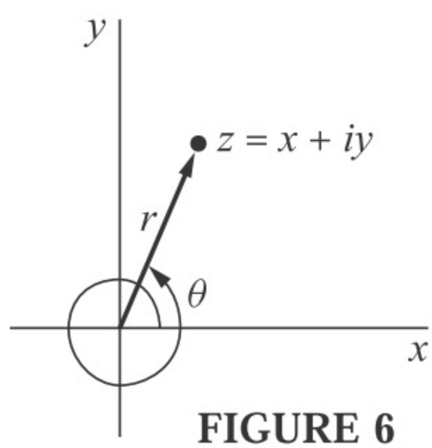

FIGURE 6

Note. We then have for any $z \in \mathbb{C}$, $z \neq 0$, that $\arg(z) = \mathrm{Arg}(z) + 2n\pi$ where $n \in \mathbb{Z}$. The choice of the principal value of the argument is not universal and it is sometimes chosen to be in $[0, 2\pi)$. This will have an effect when we define principal square roots, for example.

Definition. We define *Euler's formula* as $e^{i\theta} = \cos\theta + i\sin\theta$.

Note. The polar form of $z \in \mathbb{C}$, $z \neq 0$, can be expressed using Euler's formula as $z = re^{i\theta}$ where $r = |z|$ and $\theta \in \arg(z)$. This is called the *exponential form* of z. For now, Euler's formula is just a notation. In Section 29, it will be better motivated. For now, consider the following argument.

Recall the power series

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.
\end{aligned}$$

We have

$$e^{i\theta} = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} = \sum_{\substack{n=0 \\ n\equiv 0 (\mathrm{mod}\ 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n\equiv 1 (\mathrm{mod}\ 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n\equiv 2 (\mathrm{mod}\ 4)}}^{\infty} \frac{(i\theta)^n}{n!} + \sum_{\substack{n=0 \\ n\equiv 3 (\mathrm{mod}\ 4)}}^{\infty} \frac{(i\theta)^n}{n!}$$

since the series converges absolutely

$$= \sum_{\substack{n=0 \\ n\equiv 0 (\mathrm{mod}\ 4)}}^{\infty} \frac{(\theta)^n}{n!} + \sum_{\substack{n=0 \\ n\equiv 1 (\mathrm{mod}\ 4)}}^{\infty} i\frac{\theta^n}{n!} + \sum_{\substack{n=0 \\ n\equiv 2 (\mathrm{mod}\ 4)}}^{\infty} \left(-\frac{\theta^n}{n!}\right) + \sum_{\substack{n=0 \\ n\equiv 3 (\mathrm{mod}\ 4)}}^{\infty} \left(-i\frac{\theta^n}{n!}\right)$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta.$$

We have not defined "absolute convergence" nor do we have any theory of series in the complex setting. Once we do, this computation can be justified.

Note. The set of complex numbers of the form $z = e^{i\theta}$ is the set of all complex numbers of modulus 1, $|z| = 1$. With $\theta = \pi$, we get the popular t-shirt equation $e^{i\pi} = -1$. Notice that we also have $e^{i\pi/2} = i$ and $e^{i3\pi/2} = -i$. See Figure 7.

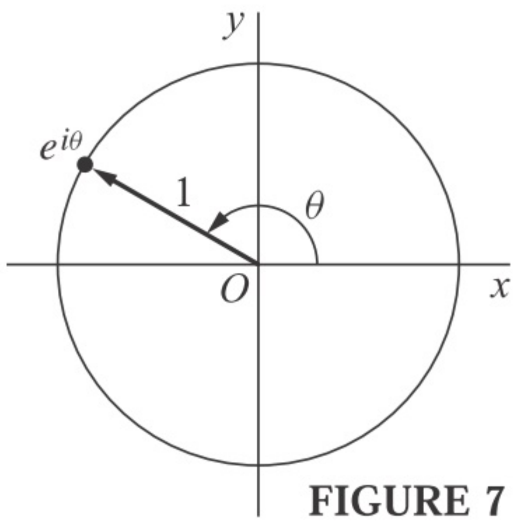

FIGURE 7

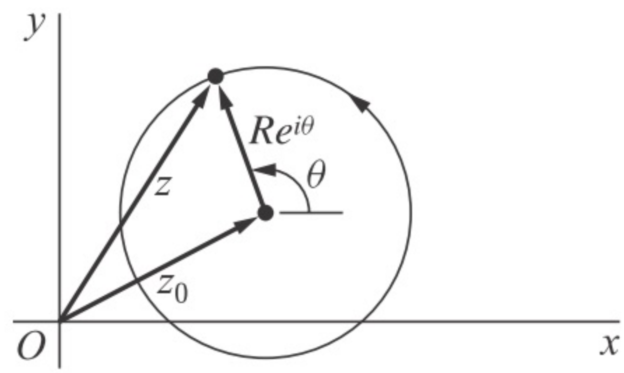

FIGURE 8

Note. In general, the circle $\{z \mid z \in \mathbb{C}, |z| = R\}$ can be expressed in exponential form as $z = Re^{i\theta}$ where θ ranges over $(-\pi, \pi]$. The circle with center z_0 and radius R is then of the form $z = z_0 + Re^{i\theta}$ where $\theta \in (-\pi, \pi]$. See Figure 8.

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