Section 1.7. Products and Powers in Exponential Form

Note. In this section, we use the exponential form to consider products, quotients, and integer powers of complex numbers. For now, the notation $z = re^{i\theta}$ is just shorthand notation for the polar representation of complex number $z = r(\cos\theta + i\sin\theta)$. We'll explore the exponential function in the complex setting in Section 2.14. Mappings by the Exponential Function and Section 3.29. The Exponential Function. We start with the product of two complex numbers represented in polar form.

Theorem 1.7.1. For $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2} \in \mathbb{C}$ we have

$$z_1z_2 = (r_1r_2)e^{i(\theta_1+\theta_2)} \text{ and } \frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

Note 1.7.A. The theorem implies that for $z = re^{i\theta}$ we have

$$z^{-1} = \frac{1}{z} = \frac{r_1e^{i0}}{r_2e^{i\theta}} = \frac{1}{r}e^{i0}e^{-i\theta} = \frac{1}{r}e^{-i\theta}.$$

With this observation, Theorem 1.7.1 makes it easy to raise a complex number in polar form to an integer power.

Corollary 1.7.2. If $z = re^{i\theta} \in \mathbb{C}$, then for $n \in \mathbb{Z}$ we have $z^n = r^ne^{in\theta}$.

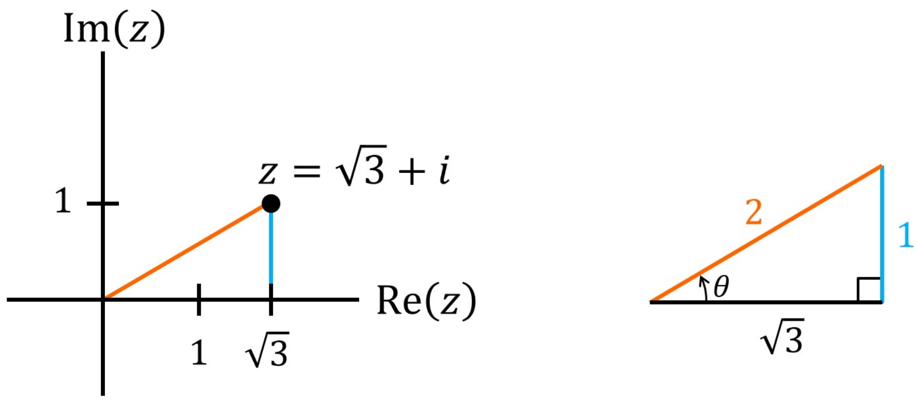
Note. Corollary 1.7.2 allows us to extract a generalization of Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$. The generalization involves $(e^{i\theta})^n$ and is called de Moivre's

Formula. This is named for Abraham de Moivre (May 26, 1667–November 27, 1754). He worked in analytic geometry and probability theory, but the de Moivre Formula makes a connection between trigonometry and analysis. He published in in the form below in 1722, but he also published a related formula in 1707. For more de Moivre, see the MacTutor biography page on de Moivre (accessed 2/3/2024).

Corollary 1.7.3. (de Moivre's Formula) For all $n \in \mathbb{Z}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example 1.7.1. We can easily compute $(\sqrt{3}+i)^7$. If we plot $z = \sqrt{3}+i$ in an Argand diagram, we see that it is in the "first quadrant," and with θ as the principal argument of z, we have $\tan\theta = 1/\sqrt{3}$. Since $0 < \theta < \pi/2$, then we can calculate θ as $\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$.



Then $\mathrm{Arg}(\sqrt{3}+i) = \pi/6$ and $r = |\sqrt{3}+i| = \sqrt{(\sqrt{3})^2+(1)^2} = 2$, so $\sqrt{3}+i = 2e^{i\pi/6}$. Hence

$$(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = (2e^{i\pi/6})^6(2e^{i\pi/6}) = (2^6e^{6i\pi/6})(2e^{i\pi/6}) = (2^6e^{i\pi})(2e^{i\pi/6})$$

$$= (64)(-1)\left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right) = (-64)(2)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -64(\sqrt{3}+i).$$

That is, $(\sqrt{3}+i)^7 = -64(\sqrt{3}+i)$.

Revised: 2/3/2024