Section 1.8. Arguments of Products and Quotients

Note. We now use the results of the previous section to relate the sets $\arg(z_1)$ and $\arg(z_2)$ with the sets $\arg(z_1 z_2)$, $\arg(z_1/z_2)$, and $\arg(1/z_2)$.

Note. For $z_1 = r_1 e^{i\theta}$ and $z_2 = r_2 e^{i\theta_2}$, we have by Theorem 1.7.1 that $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$. We can therefore say that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Brown and Churchill say (see page 20): "This result is to be interpreted as saying that if values of two of the three (multivalued) arguments are specified, then there is a value of the third such that the equation holds." This is consistent with the interpretation of the sum of two sets of real numbers represents the set consisting of all possible sums of the two set: $A + B = \{a + b \mid a \in A, b \in B\}$. We state the equation as a lemma.

Lemma 1.8.1. For any $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq 0$, $z_2 \neq 0$, we have $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Lemma 1.8.2. For any $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq 0$, $z_2 \neq 0$ we have $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.

Note. Of course, Lemmas 1.8.1 and 1.8.2 do not hold for the principal argument function, $\mathrm{Arg}(z)$.

Example 1.8.2. For $z = -2/(1 + \sqrt{3}i)$, we have by Lemma 1.8.2, $\arg(z) = \arg(-2) - \arg(1 + \sqrt{3}i)$. An argument of -2 is π and an argument of $1 + \sqrt{3}i$ is $\tan^{-1}(\sqrt{3}/1) = \tan^{-1}(\sqrt{3}) = \pi/3$ (since $1+\sqrt{3}i$ lies in the "first quadrant," we can use $\tan^{-1}$ to find an argument; this would also work for a complex number in the "fourth quadrant"). So an argument of $z = -2/(1+\sqrt{3}i)$ is $\pi - \pi/3 = 2\pi/3$. Since the principal argument is between $-\pi$ and π, we have $\mathrm{Arg}(-2/(1 + \sqrt{3}i)) = 2\pi/3$.

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