Section 1.9. Roots of Complex Numbers

Note. We now use the results of the previous two sections to find nth roots of complex numbers. In $\mathbb{R}$, there are two "choices" for a square root of x when $x > 0$ (a positive square root and a negative square root). This problem is compounded in the complex setting by the fact that there are n "choices" for the nth root of a nonzero complex number.

Note. You may have seen "nth roots of unity" in Introduction to Modern Algebra (MATH 4127/5127; see my online class notes on I. Groups and Subgroups, Section 1. Introduction and Examples.). The nth roots of unity form a cyclic group of order n under multiplication.

Note 1.9.A. Since the function $e^{i\theta}$ is a periodic function in θ with period 2π (in fact, as a function of θ graphed in the complex plane, this function traces out the unit circle $|z| = 1$). So if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ (where $r_1 > 0$ and $r_2 > 0$) then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$.

Note 1.9.B. Suppose $z_0 = r_0 e^{i\theta_0}$ and $z^n = z_0$ where $z = re^{i\theta}$. Then it must be that $z^n = r^n e^{in\theta} = z_0 = r_0 e^{i\theta_0}$ and so $r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$ for some $k \in \mathbb{Z}$. So we must have $r = \sqrt[n]{r_0}$ and $\theta = (\theta_0 + 2k\pi)/n$ for $k \in \mathbb{Z}$. Therefore, the nth roots of z_0 are $z = \sqrt[n]{r_0} \exp(i(\theta_0 + 2k\pi)/n)$ for $k \in \mathbb{Z}$. However, since $\exp(i\theta)$ is periodic, there are in fact only n distinct nth roots of z_0. Namely

$$c_k = \sqrt[n]{r_0} \exp\left(i\frac{\theta_0 + 2k\pi}{n}\right) \text{ for } k = 0, 1, \ldots, n-1.$$

With $n \geq 3$, the roots lie at the vertices of a regular n-gon inscribed in a circle of radius $\sqrt[n]{r}$ and centered at 0 (we'll have illustrations of this in the next section). When θ_0 is the principal argument of z then c_0 is the *principal nth root* of z.

Note 1.9.C. If $z = 1 = 1e^{i0}$, we get the "nth roots of unity"

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \text{ for } k = 0, 1, \ldots, n-1.$$

Notice that $\omega_n^1 = \exp(i2\pi/n)$ can be used to generate each of the other nth roots of unity: $\omega_n^k = (\omega_n^1)^k$. This is how we can form a cyclic group out of the nth roots of unity and ω_n^1 is a generator of this cyclic group (which is isomorphic to $\langle \mathbb{Z}_n, +\rangle$).

Revised: 1/29/2024