# Chapter 10. Applications of Conformal Mapping

**Note.** In this chapter, we consider applications of the conformal mappings introduced in Chapter 9. We use such mappings to solve applied problems involving Laplace's equation, which we stated in Section 2.26. Harmonic Functions (namely, $H_{xx}(x, y) + H_{yy}(x, y) = 0$). We consider applications involving temperature, electrostatic potential, and fluid flow.

## Section 118. Steady Temperatures

**Note.** In this section, we consider the amount of heat energy (as measured by temperature) distributed in a two-dimensional plate when it has reached an equilibrium. In Applied Mathematics 2 (MATH 5620), or maybe a senior level partial differential equation ("PDE") class, you might see that the temperature distribution $T(x, y, t)$ at point $(x, y)$ at time $t$ satisfies $c\rho T_t = \kappa(T_{xx} + T_{yy})$ where $c$ is the specific heat and $\rho$ is the density of the material involved (both constants), and $\kappa$ is the heat conductivity (a constant) of the material. See my online notes for Applied Mathematics 2 on Section 1.3. Flows, Vibrations, and Diffusions for a derivation of this PDE in the three-dimensional setting (the derivation there concerns the more physically realistic three spatial dimensions case). Since we are considering "steady temperatures," the we have no change of $T(x, y)$ with time, so the PDE simplifies to Laplace's equation: $T_{xx}(x, y) + T_{yy}(x, y) = 0$.

**Note/Defintion.** Brown and Churchill also describe a three-dimensional heat conduction problem, but then assume no dependence in the third dimension. The flow of heat across a surface in a solid body at a point on the surface is called *flux*. The quantity of heat that flows in a specified direction normal to the surface per unit time per unit area has units of, for example, calories/(second$^2$ cm$^2$)). We denote flux as $\Phi$. With $N$ as a normal vector to the surface (as a function of the coordinates of the points on the surface), we have that the flux satisfies $\Phi = -K\dfrac{dT}{dN}$, where $K$ is a positive constant called the *thermal conductivity* of the material involved. This relationship is called *Fourier's Law.*

**Note.** As Brown and Churchill state on page 366:

> "...we restrict out attention to those cases in which the temperature $T$ varies with only the $x$ and $y$ coordinates. ...the flow of heat is, then, two dimensional and parallel to [the plane containing the plate]. We agree, moreover, that the flow is in a steady state; that is, $T$ does not vary with time."

In addition, we assume that there are no heat sources or sinks within the plate. The temperature function $T(x, y)$ and its first and second partial derivatives are assumed to be continuous on the interior of the plate. In this way, the equilibrium temperature distribution $T(x, y)$ will be determined by Laplace's equation (i.e., the PDE) and the temperature along the boundary (i.e., the boundary values).

**Note 118.A.** We now give a heuristic argument that Laplace's equation gives a condition on the temperature distribution function $T(x, y)$ when the temperature has reached equilibrium. Consider a small rectangular area of dimensions $\Delta x$ by $\Delta y$ in the region on which $T(x, y)$ is defined; see Figure 154.

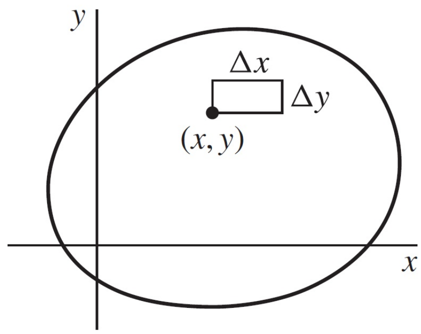


**FIGURE 154**

The flow of heat through the left-hand side of the rectangle, by Fourier's law, is $-KT_x(x, y)\Delta y$ where $K$ is the thermal conductivity, $T_x(x, y)$ is the rate of change of $T$ in the positive $x$ direction (at the point $(x, y)$; we are using this value to *approximate* the flow of temperature across all of the left-hand edge), and $\Delta y$ is the length of the left-hand edge. Similarly, the flow of heat through the right-hand side of the rectangle (a distance of $\Delta x$ from the left-hand side) is $= KT_x(x+\Delta x, y)\Delta y$. The first term represents heat entering the rectangle from the left and the second term represents heat leaving the rectangle towards the right (though these could be negative themselves), so the net rate of heat *loss* through the two vertical sides of the rectangle is the second term minus the first:

$$(-KT_x(x+\Delta x, y)\Delta y) - (-KT_x(x, y)\Delta y) = -K\left[\frac{T_x(x+\Delta x, y) - T_x(x, y)}{\Delta x}\right]\Delta x\Delta y.$$

A similar argument involving the top and bottom sides of the rectangle gives that the net rate of heat loss through these sides is

$$(-KT_y(x, y+\Delta y)\Delta x) - (-KT_y(x, y)\Delta x) = -K\left[\frac{T_y(x, y+\Delta y) - T_y(x, y)}{\Delta y}\right]\Delta x\Delta y.$$

Since we are considering a steady temperature case (when temperature equilibrium has been reached in the plate), the total net flow of heat must be 0. That is,

$$\left(-K\left[\frac{T_x(x+\Delta x,y)-T_x(x,y)}{\Delta x}\right]\Delta x\Delta y\right)$$
$$+\left(-K\left[\frac{T_y(x,y+\Delta y)-T_y(x,y)}{\Delta y}\right]\Delta x\Delta y\right)=0,$$

or (dividing out $-K\Delta x\Delta y$)

$$\frac{T_x(x+\Delta x,y)-T_x(x,y)}{\Delta x}+\frac{T_y(x,y+\Delta y)-T_y(x,y)}{\Delta y}=0.$$

Now taking limits as $\Delta x \to 0$ and $\Delta y \to 0$, we get second partial derivatives and Laplace's formula: $T_{xx}(x,y)+T_{yy}(x,y)=0$. Notice that this must hold on the interior of the region defining the plate. We now see that $T(x,y)$ must be a harmonic function on the interior of the region.

**Definition.** The level curves $T(x,y)=c_1$ where $c_1$ is any real constant (Brown and Churchill calls these "surfaces") are *isotherms*.

**Note/Definition.** As seen in Calculus 3 (MATH 2110), the gradient of function $f(x,y)$ at a point $(x_0,y_0)$ is normal to the level curve of $f$ which passes through $(x_0,y_0)$. See my online notes for that class on Section 14.5. Directional Derivatives and Gradient Vectors and notice Theorem 14.5.B. For $T(x,y)$ satisfying Laplace's equation and $S(x,y)$ a harmonic conjugate of $T$, the level curve $S(x,y)=c_2$ has the gradient of $T$ as a tangent vector at each point where the analytic function $T(x,y)+iS(s,y)$ is conformal (as is shown in Exercise 2.27.2). So level curve $S(x,y)=c_2$ is called a *line of flow*, since it reflects the path along which heat flows.

**Note 118.B.** If the normal derivative $dT/dN$ is zero along part of the boundary of the plate, then the flux of heat across that part of the boundary is zero. In this case, that part of the boundary is insulated and that part of the boundary is a line of flow.

*Revised: 2/22/2024*