Section 119. Steady Temperatures in a Half Plane

Note. In this section and the following two, we consider specific boundary value problems based on Laplace's equation and a steady (i.e., equilibrium) temperature distribution.

Note. Consider a plate in the shape of an infinite half plane, say lying in the upper half plane (where $y \geq 0$) of the complex plane. Suppose the edge $y = 0$ is perfectly insulated, that the temperature is a constant of 1 on the segment $-1 < x < 1$, and the temperature is a constant of 0 on the segments $(-\infty, -1) \cup (1, \infty)$. See Figure 155.

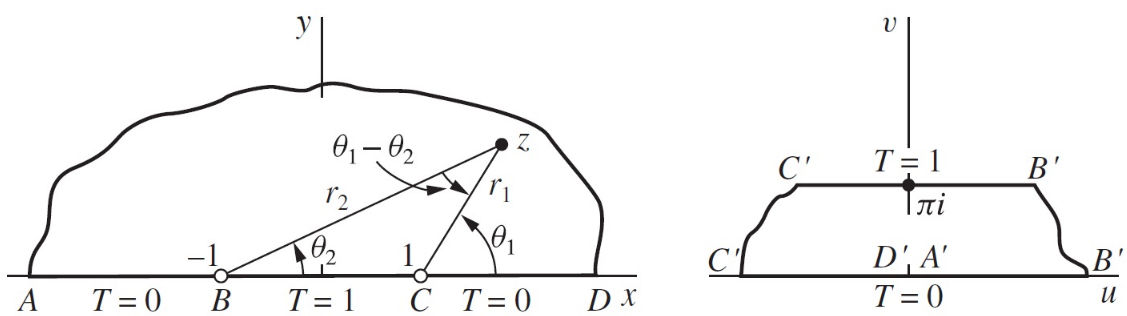


FIGURE 155 $w = \log \dfrac{z-1}{z+1} \left(\dfrac{r_1}{r_2} > 0, -\dfrac{\pi}{2} < \theta_1 - \theta_2 < \dfrac{3\pi}{2} \right)$.

We let $T(x, y)$ be the temperature at point (x, y) when the temperature is at equilibrium. The boundary problem we are faced with is to find $T(x, y)$ satisfying Laplace's equation on the open upper half plane (where $y > 0$) and satisfying the boundary conditions. We also impose a condition of boundedness on $T(x, y)$ (a reasonable physical constraint). That is:

$$T_{xx}(x, y) + T_{yy}(x, y) = 0 \text{ for } -\infty < x < \infty,\ y > 0 \text{ and}$$

$$T(x,0) = \begin{cases} 1 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1, \end{cases}$$

and $|T(x,y)| < M$ for some positive constant M. Notice this is a Dirichlet problem (as defined in Section 116. Transformations of Harmonic Functions).

Note. We'll solve the above Dirichlet problem by transforming it into a region in the uv plane that is the image under some transformation $w = f(z)$ of the region given in Figure 155 (left), where f is analytic in the upper half-plane and conformal on the boundary $y = 0$ (except at the point $(-1,0)$ and $(1,0)$ where f is undefined). The problem affiliated with the region in the uv plane (or the "w plane") will be easily solved. We then use Theorems 116.A and 117A to "pull back" the solution from the uv plane to the z plane. Theorem 116.A guarantees that the harmonic function in the w plane pulls back to a harmonic function in the z plane, and Theorem 117.A guarantees that the boundary conditions in the w plane pull back to the appropriate boundary conditions in the z plane. We start by browsing Appendix A, "Table of Transformations of Regions," and look for a transformation that maps the upper half plane to a "convenient" region. Notice Figure 19 below from the Appendix A.

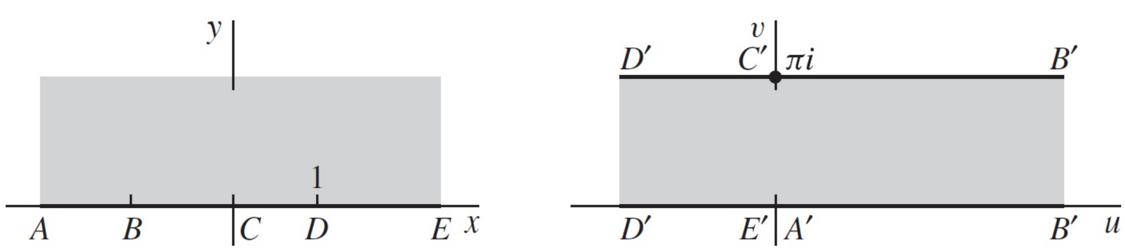


FIGURE 19 $w = \text{Log}\,\dfrac{z-1}{z+1}$; $z = -\coth\dfrac{w}{2}$.

This suggest that we consider the transformation $w = f(z) = \text{Log}\dfrac{z-1}{z+1}$. Notice this maps the interval $(-1,1) = (B,D)$ on the x axis to the horizontal line $v = i\pi$, maps interval $(1,\infty) = (D,\infty)$ on the x axis to the interval $(-\infty,0)$ on the u axis, and maps the interval $(-\infty,-1) = (-\infty,B)$ on the x axis to the interval $(0,\infty)$ on the u axis. To better visualize this, think of $x = 1 = D$ as mapped to $-\infty$ (to the far left in the W plane) and $x = -1 = B$ mapped to ∞ (to the far right in the w plane). Since $w = f(z) = \text{Log}\dfrac{z-1}{z+1}$ is analytic on the (open) upper half plane and conformal on the boundary (except at ± 1) because it has nonzero derivative there, then the boundary conditions transform from the z plane to the w plane by Theorem 117.A. Notice this means that the boundary condition $T = 1$ in the z plane is mapped to the boundary condition $T = 1$ along the line $y = i\pi$ in the w plane, and the boundary condition $T = 0$ is mapped to the boundary condition $T = 0$ along the u axis in the w plane (as shown in Figure 155, right). We need a harmonic function on the region of the w plane as given in Figure 155 (right) which satisfies the boundary conditions given in that figure.

Note. We can easily solve the Dirichlet problem in the w plane. We take $T(u,v) = v/\pi$. This is harmonic because it is the imaginary part of the analytic function $g(w) = w/\pi$, and it clearly satisfies the boundary conditions. Since we use the principal value of the logarithm (see Section 3.30. The Logarithm Function in the notes based on the 8th edition of the book), then we have

$$w = u + iv = f(z) = \text{Log}\frac{z-1}{z+1} = \ln\left|\frac{z-1}{z+1}\right| + i\text{Arg}\left(\frac{z-1}{z+1}\right).$$

Therefore $v = \text{Arg}\left(\dfrac{z-1}{z+1}\right)$ and $v/\pi = \dfrac{1}{\pi}\text{Arg}\left(\dfrac{z-1}{z+1}\right)$. We just need to convert this into xy coordinates. With $z = x + iy$ we have

$$T(x, y) = \frac{1}{\pi}\text{Arg}\left(\frac{(x + iy) - 1}{(x + iy) + 1}\right).$$

Since

$$\frac{(x + iy) - 1}{(x + iy) + 1} = \frac{(x - 1) + iy}{(x + 1) + iy} = \left(\frac{(x - 1) + iy}{(x + 1) + iy}\right)\left(\frac{(x + 1) - iy}{(x + 1) - iy}\right)$$

$$= \frac{(x^2 - 1) + y^2 + i(y(x + 1) - y(x - 1))}{(x + 1)^2 + y^2} = \frac{x^2 + y^2 - 1 + i(2y)}{x^2 + 2x + y^2 + 1},$$

then

$$T(x, y) = \frac{1}{\pi}\text{Arg}\left(\frac{x^2 + y^2 - 1 + i(2y)}{x^2 + 2x + y^2 + 1}\right).$$

Notice that in the open upper half plane, the principal argument will be (strictly) between 0 and π. This is the solution to the given Dirichlet problem.

Note. To further explore the solution, we modify it and express it in terms of the real arctangent function. Recall that for $z = x + iy = re^{i\theta}$, where $x \neq 0$, we have $\tan\theta = y/x$. Brown and Churchill also use the arctangent function, but we are more careful than they are. Recall that the range of the arctangent function is $(-\pi/2, \pi/2)$ (see my online Calculus 1 notes on Section 1.6. Inverse Functions and Logarithms, for example). However, the principal argument function will return values over the upper half plane which are between 0 and π. So we cannot (as Brown and Churchill do) simply convert $T(x, y)$ into terms of the real arctangent function. We *can* directly use the arctangent function in the open first quadrant (where principal arguments are strictly between 0 and $\pi/2$, and we can indirectly use it in the open second quadrant (where principal arguments are strictly between $\pi/2$ and π). However, we cannot use it when the principal argument is $\pi/2$ (which is the case along the positive y axis). This leads us to express $T(x,y)$ piecewise in terms arctangent as follows:

$$
\begin{aligned}
T(x,y) &= \frac{1}{\pi}\mathrm{Arg}\left(\frac{x^2+y^2-1+i(2y)}{x^2+2x+y^2+1}\right) \\
&= \frac{1}{\pi}\mathrm{Arg}\left(\frac{x^2+y^2-1}{x^2+2x+y^2+1}+i\frac{2y}{x^2+2x+y^2+1}\right) \\
&= \begin{cases} \frac{1}{\pi}\arctan\left(\frac{2y}{x^2+y^2-1}\right) & \text{if } \frac{2y}{x^2+y^2-1}>0 \\ 1/2 & \text{if } x^2+y^2=1 \\ \frac{1}{\pi}\left(\arctan\left(\frac{2y}{x^2+y^2-1}\right)+\pi\right) & \text{if } \frac{2y}{x^2+y^2-1}<0 \end{cases} \\
&= \begin{cases} \frac{1}{\pi}\arctan\left(\frac{2y}{x^2+y^2-1}\right) & \text{if } x^2+y^2>1 \\ 1/2 & \text{if } x^2+y^2=1 \\ \frac{1}{\pi}\left(\arctan\left(\frac{2y}{x^2+y^2-1}\right)+\pi\right) & \text{if } x^2+y^2<1 \end{cases}
\end{aligned}
$$

where the last equality holds because the sign of $2y/(x^2+y^2-1)$ is the same as the sign of x^2+y^2-1, since $y>0$ in the upper half plane of the z plane.

Note. With an explicit expression for $T(x,y)$, we can verify directly that it is harmonic on the upper half plane and that it satisfies the boundary conditions. For example, for a point on the real axis of the form $(x_B,0)$ where $-1<x_B<1$, we can take a limit as $(x,y)\to(x_B,0)$ for (x,y) in the upper half plane. Notice that we can assume that (x,y) has coordinates satisfying $x^2+y^2<1$ and, since arctan is continuous at 0, we have

$$
\lim_{(x,y)\to(x_B,0)}\frac{1}{\pi}\left(\arctan\left(\frac{2y}{x^2+y^2-1}\right)+\pi\right)
$$

$$
=1+\lim_{(x,y)\to(x_B,0)}\arctan\left(\frac{2y}{x^2+y^2-1}\right)=1+\arctan(0)=1+0=1,
$$

as expected. Similarly, if $(x, y) \to (x_B, 0)$ for (x, y) in the upper half plane for $|x_B| > 1$, $T(x, y)$ has the limit value of 0.

Note. For isotherms of $T(x, y)$ we set $T(x, y) = c_1$, where c_1 is a constant satisfying $0 < c_1 < 1$ (notice that $T(x, y)$ is always between 0 and 1). First, notice that for $c_1 = 1/2$, the isotherms are the upper half of the circle $x^2 + y^2 = 1$. More generally,

$$\frac{1}{\pi} \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) = c_1 \text{ implies } \frac{2y}{x^2 + y^2 - 1} = \tan(\pi c_1)$$

or $2y = (x^2 + y^2 - 1)\tan(\pi c_1)$ or $2y \cot(\pi c_1) = x^2 + y^2 - 1$ or $x^2 + y^2 - 2y \cot(\pi c_1) = 1$ or, since $1 + \cot^2 \theta = \csc^2 \theta$, $x^2 + y^2 - 2y \cot(\pi c_1) = \csc^2(\pi c_1) - \cot^2(\pi c_1)$ or $x^2 + y^2 - 2y \cot(\pi c_1) + \cot^2(\pi c_1) = \csc^2(\pi c_1)$ or

$$x^2 + (y - \cot \pi c_1)^2 = \csc^2(\pi c_1).$$

That is, isotherms are arcs of circles of the form $x^2 + (y - \cot(\pi c_1))^2 = \csc^2(\pi c_1)$; the arcs are contained in the upper half plane. These circles have centers $(0, \cot(\pi c_1))$ on the y axis and they pass through the points $(-1, 0)$ and $(1, 0)$ (since $(x, y) = (\pm 1, 0)$ reduces the equation of the isotherm to the identity $1 + \cot^2(\pi c_1) = \csc^2(\pi c_1)$). Similarly,

$$\frac{1}{\pi}\left(\arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) + \pi\right) = c_1$$

implies

$$\tan\left(\arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) + \pi\right) = \tan(\pi c_1)$$

or

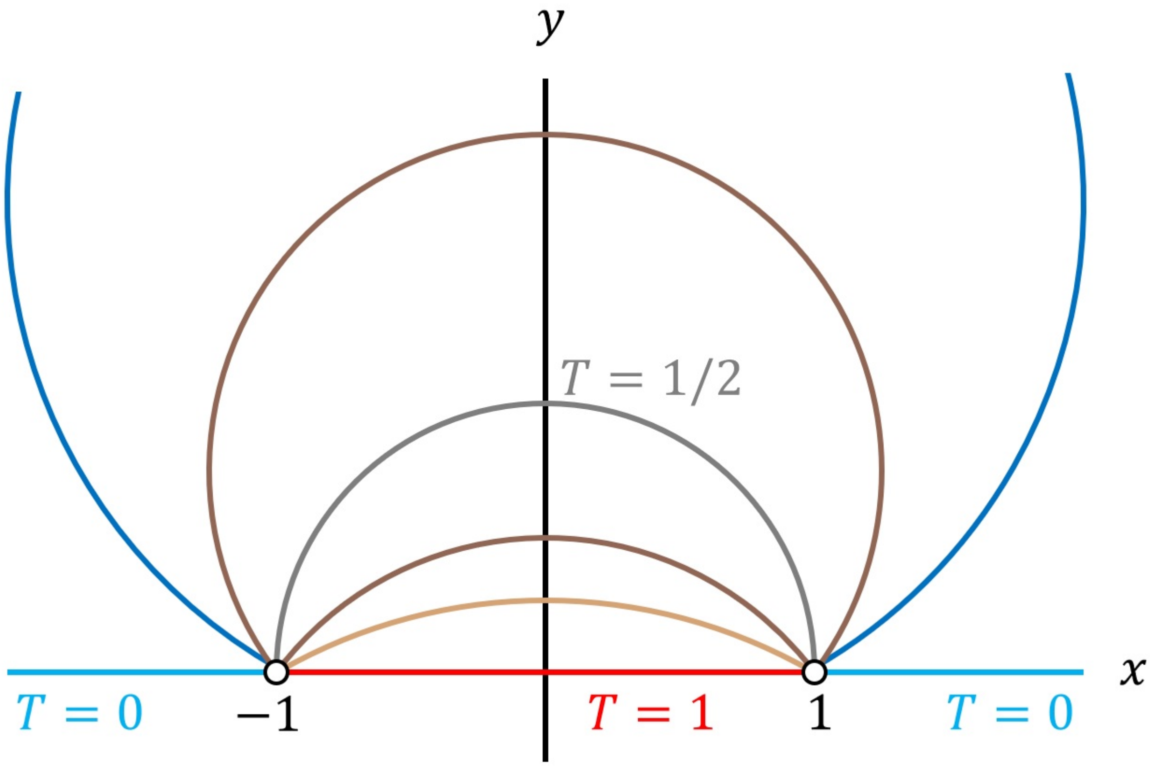
$$\tan\left(\arctan\left(\frac{2y}{x^2+y^2-1}\right)+\pi\right) = \tan\left(\arctan\left(\frac{2y}{x^2+y^2-1}\right)\right) = \tan(\pi c_1)$$

where the first equality holds because the period of tangent is π. We again have

$$\frac{2y}{x^2+y^2-1} = \tan \pi c_1,$$

and isotherms are the same as above.

Note. We now draw a few of the isotherms $x^2 + (y - \cot(\pi c_1))^2 = \csc^2(\pi c_1)$. The center of such a circle is $(0, \cot(\pi c_1))$ where $0 < \pi c_1 < \pi$ (recall $0 < c_1 < 1$). Recall that $\cot\theta$ is positive for $0 < \theta < \pi/2$ and $\cot\theta$ is negative for $\pi/2 < \theta < \pi$ (see, for example, my online Calculus 1 notes on Section 1.3. Trigonometric Functions and notice Figure 1.41). So for $0 < c_1 < 1/2$, the isotherm is a circle with center $(0, y_1)$ where y_1 is positive and the circle passes through the points $(-1, 0)$ and $(1, 0)$. For $1/2 < c_1 < 1$, the isotherm is a circle with center $(0, y_1)$ where y_1 is negative and the circle passes through the points $(-1, 0)$ and $(1, 0)$. As commented above, the circle centered at $(0, 0)$ and passing through the points $(-1, 0)$ and $(1, 0)$ corresponds to the isotherm $T(x, y) = 1/2$. Notice that you can think of the boundary where $T(x, y) = 1$ as a limit (as $y_1 \to -\infty$) of the isotherms which have centers $(0, y_1)$ where y_1 is negative. You can also think of the boundary where $T(x, y) = 0$ as a limit (as $y_1 \to \infty$) of the isotherms which have centers $(0, y_1)$ where y_1 is positive.



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