Section 120. A Related Problem

Note. In this section, we consider a specific boundary value problem based on Laplace's equation and a steady (i.e., equilibrium) temperature distribution. It concerns an rectangular plate that is infinite in one direction. We use a particular transformation to translate this problem into that of the previous section.

Note. Consider a semi-infinite plate bounded by $x = -\pi/2$, $x = \pi/2$, $y = 0$, and unbounded in the positive y direction. The temperature on the boundaries $x = \pm\pi/2$ is a constant of 0 and the temperature on the boundary $y = 0$ is a constant of 1. See Figure 156.

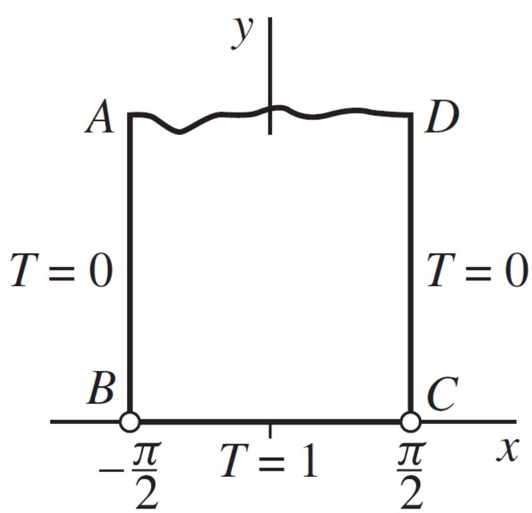


FIGURE 156

We let $T(x, y)$ be the temperature at point (x, y) when the temperature is at equilibrium. The boundary problem we are faced with is to find $T(x, y)$ satisfying Laplace's equation on the interior of the rectangle and satisfying the boundary conditions. As in the previous section, we impose a condition of boundedness on

$T(x,y)$. That is:

$$T_{xx}(x,y) + T_{yy}(x,y) = 0 \text{ for } -\pi/2 < x < \pi/2,\ y > 0,$$

$$T(-\pi/2,y) = T(\pi/2,y) = 0 \text{ for } y > 0, \text{ and } T(x,0) = 1 \text{ for } -\pi/2 < x < /\pi/2,$$

and $|T(x,y)| < M$ for some positive constant M.

Note. Browsing Appendix 2, "Table of Transformations of Regions," we notice in Figure 9 that the transformation $w = \sin z$ transforms the given region onto the region used in the example worked in Section 119. Steady Temperatures in a Half Plane.

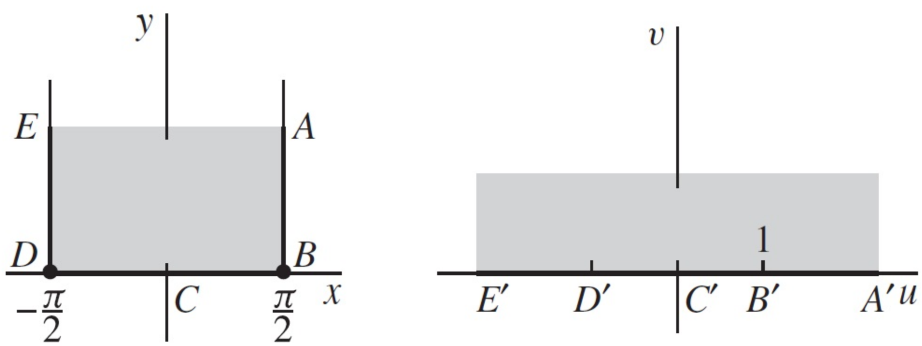


FIGURE 9
$w = \sin z$.

Also, $w = \sin z$ is analytic in the entire complex plane and, since its derivative is $\cos z$, conformal on the boundary of the region (except at $\pm\pi/2$). So with $s = \sin z$, Theorem 116.A guarantees that a harmonic function in the w plane pulls back to a harmonic function in the z plane, and Theorem 117.A guarantees that the boundary conditions in the w plane pulls back to the appropriate boundary conditions in the z plane. So in the w plane we consider the boundary problem:

$$T_{uu}(u,v) + T_{vv}(u,v) = 0 \text{ for } -1 < u < 1,\ v > 0,$$

$$T(-1,v)=T(1,v)=0 \text{ for } v>0, \text{ and } T(u,0)=1 \text{ for } -1<u<1,$$

and $|T(u,v)|<M$ for some positive constant M. We know from the previous section that the solution to *this* is:

$$T(u,v)=\begin{cases} \frac{1}{\pi}\arctan\left(\frac{2v}{u^2+v^2-1}\right) & \text{if } \frac{2v}{u^2+v^2-1}>0 \\ 1/2 & \text{if } u^2+v^2=1 \\ \frac{1}{\pi}\left(\arctan\left(\frac{2v}{u^2+v^2-1}\right)+\pi\right) & \text{if } \frac{2v}{u^2+v^2-1}<0. \end{cases}$$

We have $\sin z=\sin x\cosh y+i\cos x\sinh y$ (by Lemma 3.34.A of Section 3.34. Trigonometric Functions in the notes based on the 8th edition of Brown and Churchill), so that in the transformation $w=u+iv=\sin z$ we have $u(x,y)=\sin x\cosh y$ and $v(x,y)=\cos x\sinh y$. So the solution in the w plane pulls back to the following solution in the z plane:

$$T(u,v)=\begin{cases} \frac{1}{\pi}\arctan\left(\frac{2\cos x\sinh y}{(\sin x\cosh y)^2+(\cos x\sinh y)^2-1}\right) & \text{if } \frac{2\cos x\sinh y}{(\sin x\cosh y)^2+(\cos x\sinh y)^2-1}>0 \\ 1/2 & \text{if } (\sin x\cosh y)^2+(\cos x\sinh y)^2=1 \\ \frac{1}{\pi}\left(\arctan\left(\frac{2\cos x\sinh y}{(\sin x\cosh y)^2+(\cos x\sinh y)^2-1}\right)+\pi\right) & \text{if } \frac{2\cos x\sinh y}{(\sin x\cosh y)^2+(\cos x\sinh y)^2-1}<0. \end{cases}$$

$$=\begin{cases} \frac{1}{\pi}\arctan\left(\frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1}\right) & \text{if } \frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1}>0 \\ 1/2 & \text{if } \sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y=1 \\ \frac{1}{\pi}\left(\arctan\left(\frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1}\right)+\pi\right) & \text{if } \frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1}<0. \end{cases}$$

Using the identities $\sin^2 x=1-\cos^2 x$ and $\cosh^2 y=1+\sinh^2 y$, we have

$$\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1=(1-\cos^2 x)(1+\sinh^2 y)+\cos^2 x\sinh^2 y-1$$

$$=1+\sinh^2 y-\cos^2 x-\cos^2 x\sinh^2 y+\cos^2 x\sinh^2 y-1=\sinh^2 y-\cos^2 x,$$

so that

$$\frac{2\cos x\sinh y}{(\sin x\cosh y)^2+(\cos x\sinh y)^2-1}=\frac{2\cos x\sinh y}{\sinh^2 y-\cos^2 x}=\frac{2(\cos x/\sinh y)}{1-(\cos x/\sinh y)^2}.$$

Now the range of $\sinh y$ is $\mathbb{R}$ and we only have $\sinh y = 0$ for $y = 0$, so $\cos x/\sinh y$ could potentially be any real number. However, we also know that and $\cos x > 0$ and $\sinh y > 0$ over the region of interest, so $\cos x/\sinh y > 0$. If we set $\cos x/\sinh y = \tan\alpha$ (since the range of $\tan\alpha$ for $\alpha > 0$ is all positive real numbers, such a positive α exists for any given x and y in the region of interest), then from the double angle formula for tangent, $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$, we now have

$$\frac{2(\cos x/\sinh y)}{1-(\cos x/\sinh y)^2} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha.$$

We now have from the arctangent function that

$$T = \frac{1}{\pi}\arctan(\tan 2\alpha) = \frac{2\alpha}{\pi} = \frac{2}{\pi}\alpha,$$

where $\tan\alpha = \cos x/\sinh y$. That is,

$$T(x,y) = \frac{2}{\pi}\arctan\left(\frac{\cos x}{\sinh y}\right).$$

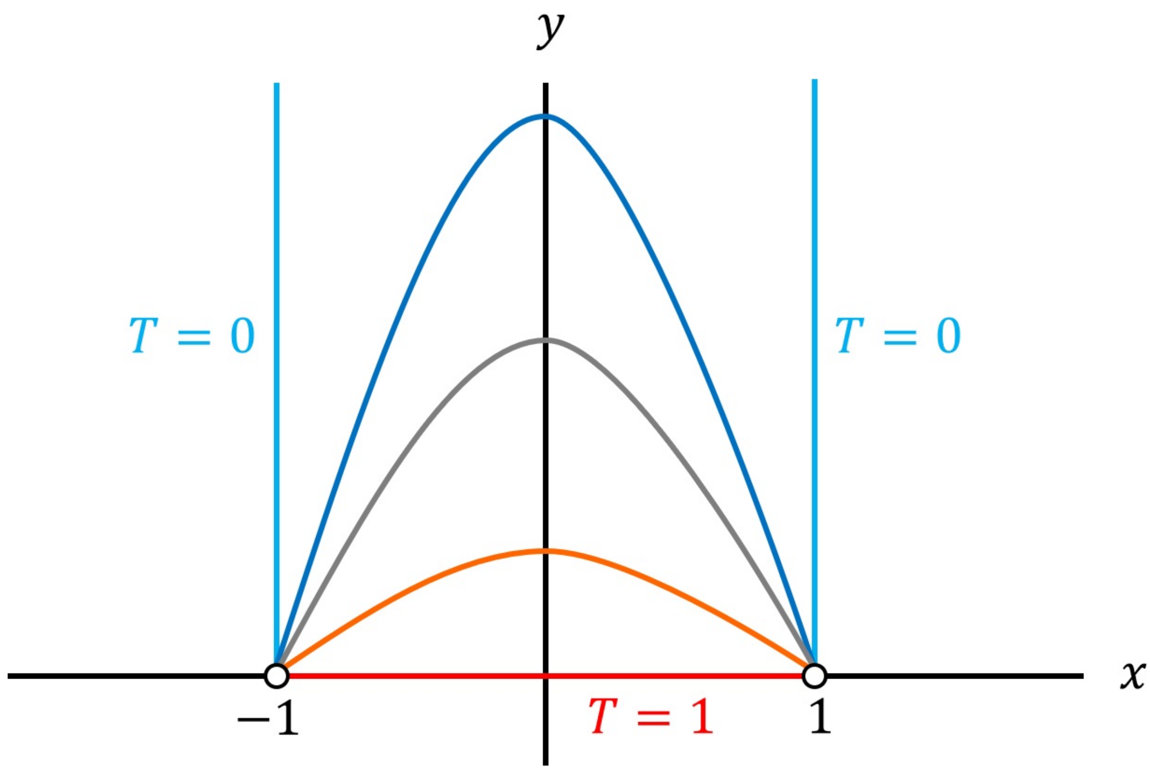
Since $\cos x/\sinh y > 0$ and we can choose α between 0 and $\pi/2$, the arctangent function outputs values between 0 and $\pi/2$, giving $T(x,y)$ values between 0 and 1, as desired (without the need for considering a piecewise definition for T). *This* is the solution to the original boundary problem.

Note. For isotherms, we consider $T(x,y) = c_1$, where $0 < c_1 < 1$. That is, we consider

$$T(x,y) = \frac{2}{\pi}\arctan\left(\frac{\cos x}{\sinh y}\right) = c_1.$$

Rearranging and taking tangents gives $\cos x = \tan(\pi c_1/2)\sinh y$. This is a highly transcendental equation, but notice that each solution passes through the the points

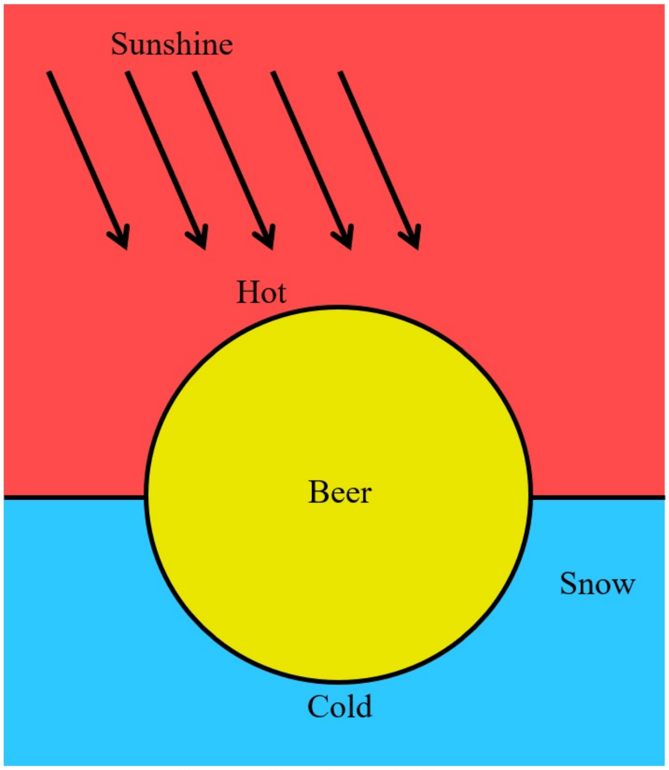
$(-\pi/2, 0)$ and $(\pi/2, 0)$ since $\cos \pi/2 = \cos(-\pi/2) = 0$ and $\sinh(0) = 0$. The graphs of the isotherms look something like the following.



Note. A related problem is the "beer in the snow" problem, given in Ralph Boas' *Invitation to Complex Analysis* (Random House, 1987) on his pages 164 and 165. The statement of the problem is:

> "...let's think of a bottle of beer lying on its side in a snowbank on a sunny day. The lower half of the bottle will be cold, the upper half will be warm, and we want to find the temperature distribution in the beer. To get a tractable mathematical model, we will think of the bottle as an infinite solid circular cylinder, with all transverse cross sections

alike, so that any one cross section will be representative. [This is sometimes called moving a boundary condition to infinity.] ... This idealization reduces the problem to a two-dimensional one, so that our model requires us (as is shown in physics) to find a harmonic function T inside a disk Δ, which we may take to be the unit disk, when the upper half of the boundary is held at temperature T_1, and the lower half at temperature T_2..."



Cross Section of Bottle

Note. The Beer in the Snow boundary problem can be stated as:

$$T_{xx}(x, y) + T_{yy}(x, y) = 0 \text{ for } x^2 + y^2 < 1$$

$$T(x, y) = T_1 \text{ for } x^2 + y^2 = 1, y > 0 \text{ and } T(x, y) = T_2 \text{ for } x^2 + y^2 = 1, y < 0.$$

Browsing Appendix 2, "Table of Transformations of Regions," for a mapping between a nice region and the unit disk, we notice Figure 13, which maps the upper half plane to the unit disk with the transformation $w = \dfrac{i-z}{i+z}$.

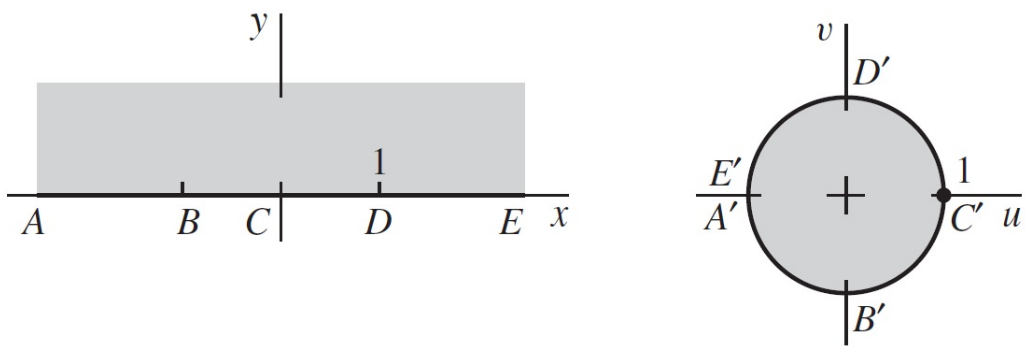


FIGURE 13

$w = \dfrac{i-z}{i+z}$.

However, we need the nice region to be an image of the disk, not the other way around. Fortunately, the mapping $w = \dfrac{i-z}{i+z}$ is a linear fractional transformation (or a "bilinear transformation") and is of the form $w = \dfrac{az+b}{cz+d}$ where $ad - bc = (-1)(i) - (i)(1) = -2i \neq 0$. As discussed in Section 8.99, "Linear Fractional Transformations" (in the 9th edition of Brown and Churchill), such a transformation has an inverse of the form $z = \dfrac{-dw+b}{cw-a}$. So the inverse transformation of $w = \dfrac{i-z}{i+z}$ is $z = \dfrac{-iw+i}{w+1}$. Interchanging z and w in Figure 13 above, gives the following transformation.

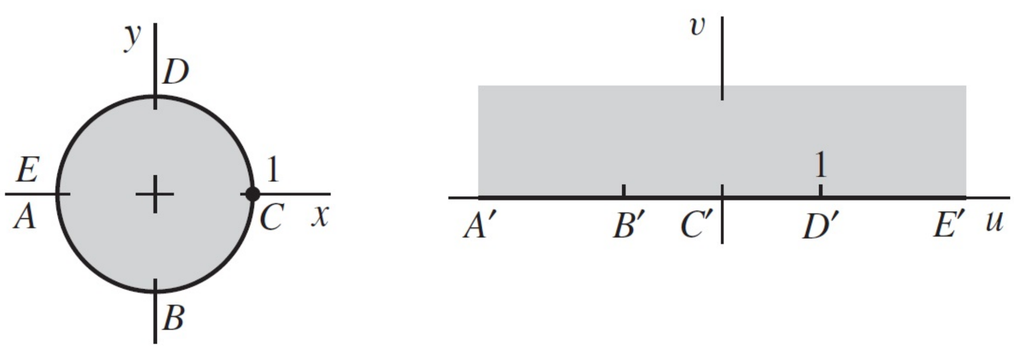


FIGURE 13′

$w = \dfrac{-iz+i}{z+1}$.

We have dealt with the upper half plane before in Section 119. Steady Temperatures in a Half Plane, but the boundary conditions are of a different nature here (being constant on the negative real axis, and constant on the positive real axis, as opposed to the condition given in Section 119 where there are three intervals on which we have constant boundary conditions). So we seek second transformation that allows us to deal with an "easy" region and "easy" boundary conditions. Notice that Figure 6 of Appendix 2 concerns a transformation between the upper half plane and an infinite horizontal strip:

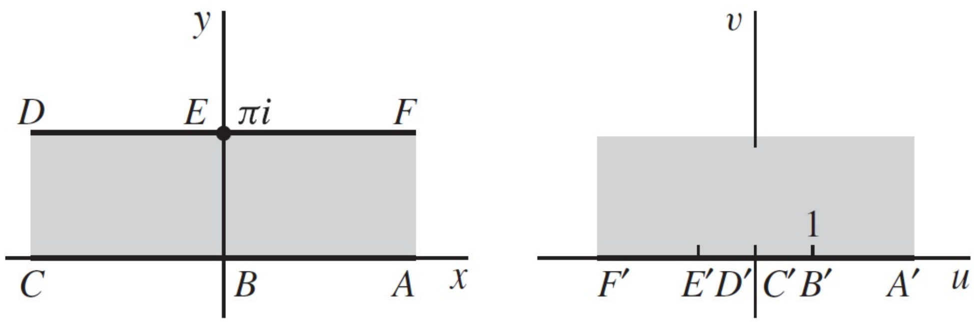


FIGURE 6
$w = \exp z$.

Again, however, we want the nice region in the w plane so that we can find a solution there and use it to "pull back" to a solution in the z plane. The mapping of Figure 6 is $w = \exp z$, which has as an inverse the principal branch of the logarithm, $z = \text{Log}\, w$. In particular, the principal branch of the logarithm returns as argument in the interval $(-\pi, \pi]$ so that it maps the upper half plane onto the horizontal strip. Interchanging z and w to consider the transformation $w = \text{Log}\, z$ gives:

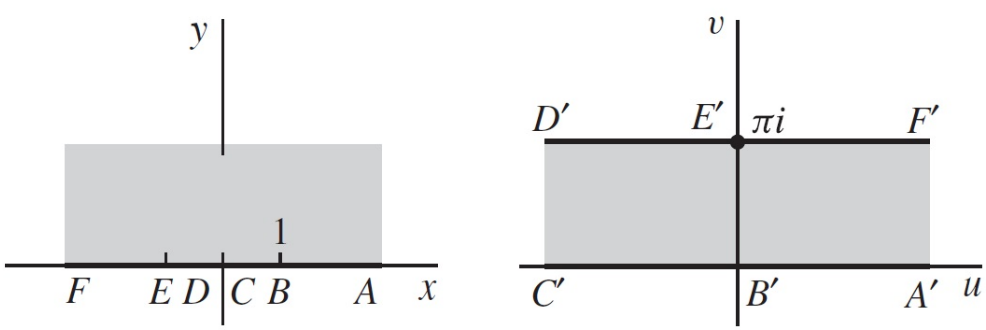
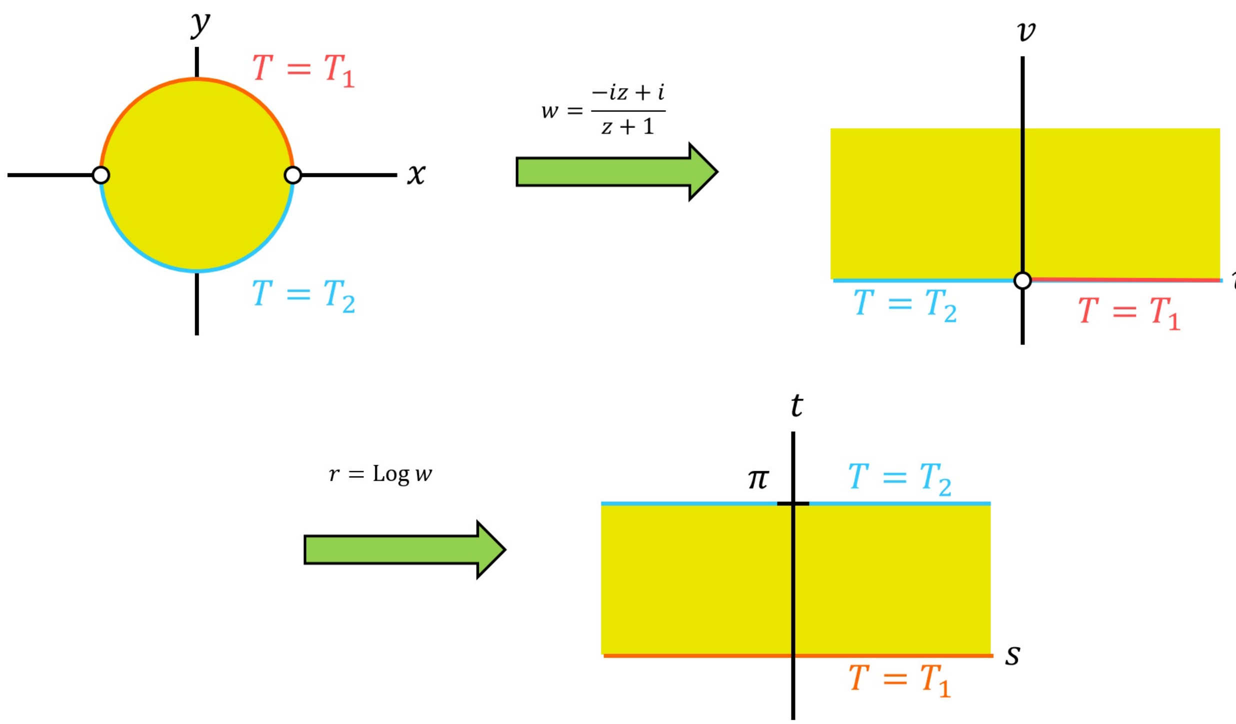


FIGURE 6′
$w = \text{Log}\, z$.

We can now compose the transformations $z = \operatorname{Log} z$ and $z = \dfrac{-iw+i}{w+1}$ to get a mapping of the horizontal strip to the upper half plane and then to the unit disk. The mappings and boundary conditions are as follows:



Here with the second transformation, we introduce the r plane, where $r = s + it$.

Note. Similar to the solution given in Section 119. Steady Temperatures in a Half Plane, we can take

$$T(s,t) = \left(\frac{T_2 - T_1}{\pi}\right) t + T_1.$$

Since $\left(\dfrac{T_2 - T_1}{\pi}\right) t$ is the imaginary part of the analytic function $g(r) = r(T_2 - T_1)/\pi$, then it is harmonic and, since adding a constant to a harmonic function yields a harmonic function, $T(s,t)$ is harmonic, and it clearly satisfies the boundary conditions. Now the composition $\operatorname{Log}\left(\dfrac{-iz+i}{z+1}\right)$ maps the unit disc to the infinite horizontal strip, as shown above. Next,

$$\text{Log}\left(\frac{-iz+i}{z+1}\right) = \ln\left|\frac{-iz+i}{z+1}\right| + i\text{Arg}\left(\frac{-iz+i}{z+1}\right) = s+it,$$

so we have $t = \text{Arg}\left(\frac{-iz+i}{z+1}\right)$. To convert this to xy coordinates, we set $z = x+iy$, from which we have

$$\frac{-iz+i}{z+1} = \left(\frac{-iz+i}{z+1}\right)\left(\frac{\bar{z}+1}{\bar{z}+1}\right) = \frac{i(-z+1)(\bar{z}+1)}{|z|^2+\bar{z}+z+1} = \frac{i(-|z|^2+\bar{z}-z+1)}{|z|^2+2\text{Re}(z)+1}$$
$$= \frac{i(-|z|^2-2i\text{Im}(z)+1)}{|z|^2+2\text{Re}(z)+1} = \frac{2y+i(-(x^2+y^2)+1)}{x^2+y^2+2y+1},$$

so

$$T(x,y) = \frac{T_2-T_1}{\pi}\text{Arg}\left(\frac{2y+i(-(x^2+y^2)+1)}{x^2+y^2+2y+1}\right)+T_1$$

is the solution of the Beer in the Snow problem.

Note. As in Section 119. Steady Temperatures in a Half Plane, we introduce the arctangent function to more clearly express $T(x,y)$ and to find isotherms. We have

$$\begin{aligned}
T(x,y) &= \frac{T_2-T_1}{\pi}\text{Arg}\left(\frac{2y+i(-(x^2+y^2)+1)}{x^2+y^2+2y+1}\right)+T_1 \\
&= \frac{T_2-T_1}{\pi}\text{Arg}\left(\frac{2y}{x^2+y^2+2y+1}+i\frac{1-x^2-y^2}{x^2+y^2+2y+1}\right)+T_1 \\
&= \begin{cases} \frac{T_2-T_1}{\pi}\arctan\left(\frac{1-x^2-y^2}{2y}\right)+T_1 & \text{if } \frac{1-x^2-y^2}{2y} > 0 \\ (T_1+T_2)/2 & \text{if } y=0 \\ \frac{T_2-T_1}{\pi}\left(\arctan\left(\frac{1-x^2-y^2}{2y}\right)+\pi\right)+T_1 & \text{if } \frac{1-x^2-y^2}{2y} < 0 \end{cases} \\
&= \begin{cases} \frac{T_2-T_1}{\pi}\arctan\left(\frac{1-x^2-y^2}{2y}\right)+T_1 & \text{if } y > 0 \\ (T_1+T_2)/2 & \text{if } y=0 \\ \frac{T_2-T_1}{\pi}\left(\arctan\left(\frac{1-x^2-y^2}{2y}\right)+\pi\right)+T_1 & \text{if } y < 0 \end{cases}
\end{aligned}$$

where the last equality holds because the sign of $(1 - x^2 - y^2)/(2y)$ is the same as the sign of y, since $1 - (x^2 + y^2) > 0$ in the open unit disk in the z plane. Now in general, $\arctan(1/x) = \pi/2 - \arctan(x) = \pi/2 + \arctan(-x)$ for $x > 0$ and $\arctan(1/x) = -\pi/2 - \arctan(x) = -\pi/2 + \arctan(-x)$ for $x < 0$, so we can write

$$\arctan\left(\frac{1 - x^2 - y^2}{2y}\right) = \frac{\pi}{2} + \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \text{ for } y > 0$$

and

$$\arctan\left(\frac{1 - x^2 - y^2}{2y}\right) = -\frac{\pi}{2} + \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \text{ for } y < 0.$$

Then

$$\begin{aligned} T(x, y) &= \begin{cases} \frac{T_2 - T_1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{2y}{x^2+y^2-1}\right)\right) + T_1 & \text{if } y > 0 \\ (T_1 + T_2)/2 & \text{if } y = 0 \\ \frac{T_2 - T_1}{\pi}\left(-\frac{\pi}{2} + \arctan\left(\frac{2y}{x^2+y^2-1}\right) + \pi\right) + T_1 & \text{if } y < 0 \end{cases} \\ &= \begin{cases} \frac{T_2 - T_1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{2y}{x^2+y^2-1}\right)\right) + T_1 & \text{if } y \neq 0 \\ (T_1 + T_2)/2 & \text{if } y = 0 \end{cases} \\ &= \begin{cases} \frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{\pi}\arctan\left(\frac{2y}{x^2+y^2-1}\right) & \text{if } y \neq 0 \\ (T_1 + T_2)/2 & \text{if } y = 0. \end{cases} \end{aligned}$$

Note. With an explicit expression for $T(x, y)$, we can verify directly that it is harmonic on the upper half plane and that it satisfies the boundary conditions. For example, for a point (x, y) in the open unit disk that approaches a point (x_B, y_B) on the boundary of the unit disk where $y_B > 0$ we have

$$\lim_{(x,y)\to(x_B,y_B)} \left(\frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{\pi}\arctan\left(\frac{2y}{x^2 + y^2 - 1}\right)\right)$$

$$= \frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{\pi}\left(-\frac{\pi}{2}\right) = T_1$$

Similarly, if (x, y) in the open unit disk that approaches a point (x_B, y_B) on the boundary of the unit disk where $y_B < 0$, $T(x, y)$ has a limit value of $(T_1 + T_2)/2 + (T_2 - T_1)/2 = T_2$.

Note. For isotherms of $T(x, y)$ we set $T(x, y) = c_1$, where c_1 is a constant such that c_1 is between T_1 and T_2 (notice that $T(x, y)$ is always between T_1 and T_2). First, notice that for $c1 = (T_1 + T_2)/2$, the isotherm is the line segment between $(-1, 0)$ and $(1, 0)$. More generally,

$$T(x, y) = \frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{\pi}\arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) = c_1$$

implies

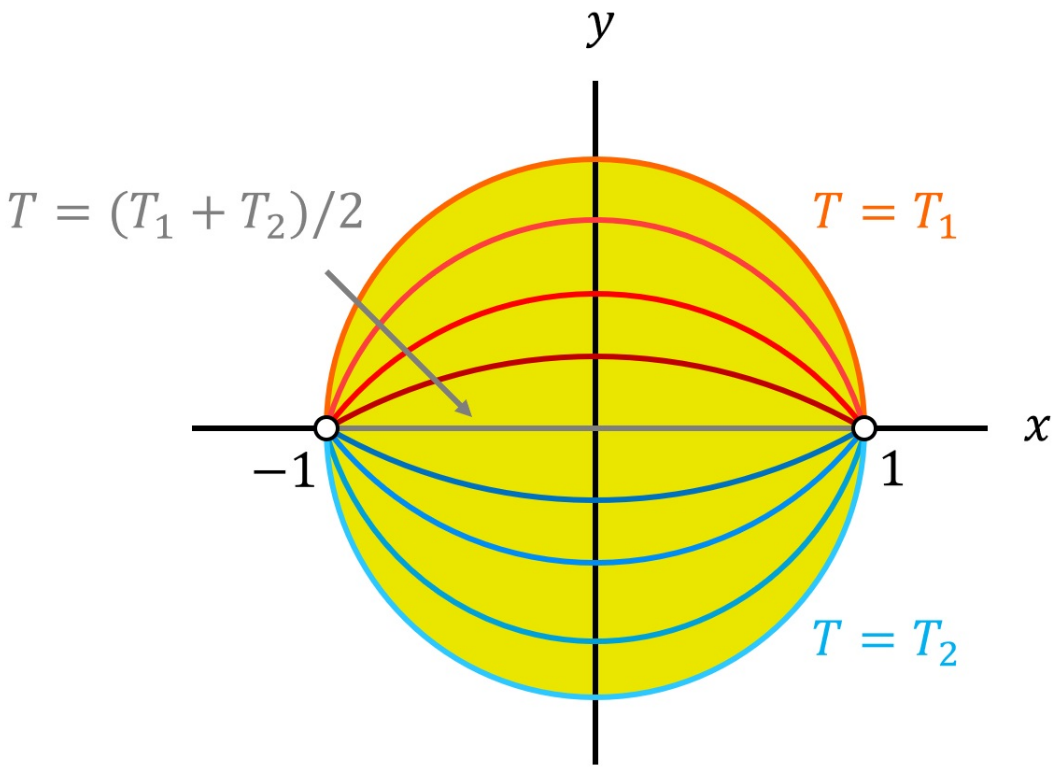
$$\arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) = \frac{\pi}{T_2 - T_1}\left(c_1 - \frac{T_1 + T_2}{2}\right) = \pi c_2$$

where $c_2 = \dfrac{1}{T_2 - T_1}\left(c_1 - \dfrac{T_1 + T_2}{2}\right)$. This gives

$$\frac{2y}{x^2 + y^2 - 1} = \tan(\pi c_2).$$

This is the same condition on the isotherms as that given in the problem solved in Section 119. Steady Temperatures in a Half Plane. So, again, the isotherms are are arcs of circles of the form $x^2 + (y - \cot(\pi c_2)^2 = \csc^2(\pi c_2)$; that are contained in the unit disk. These circles have centers $(0, \cot(\pi c_2))$ on the y axis and they pass through the points $(-1, 0)$ and $(1, 0)$ (since $(x, y) = (\pm 1, 0)$ reduces the equation of the isotherm to the identity $1 + \cot^2(\pi c_2) = \csc^2(\pi c_2)$).

Note. The isotherms are distributed similar to the following:



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