Section 121. Temperatures in a Quadrant

Note. In this section, we consider a boundary value problem involving over a quadrant of the plane which represents a temperature distribution at equilibrium where the temperature has fixed values on parts of the boundary, but the another part of the boundary is insulated (and no temperature condition is imposed on the insulated part). Being insulated, the rate of flow of hear across that part of the boundary must be 0. That is, we consider a Neumann problem as described in Section 116. Transformations of Harmonic Functions

Note. Consider an infinite plate in the plane bounded by the positive x axis and the positive y axis. At equilibrium, the temperature distribution $T(x, y)$ satisfies Laplace's equation. We require a constant temperature of 0 on the positive y axis, a constant temperature of 1 on the part of the positive x axis where $x > 1$, and the part of the positive x axis where $0 < x < 1$ is insulated. See Figure 157 (left).

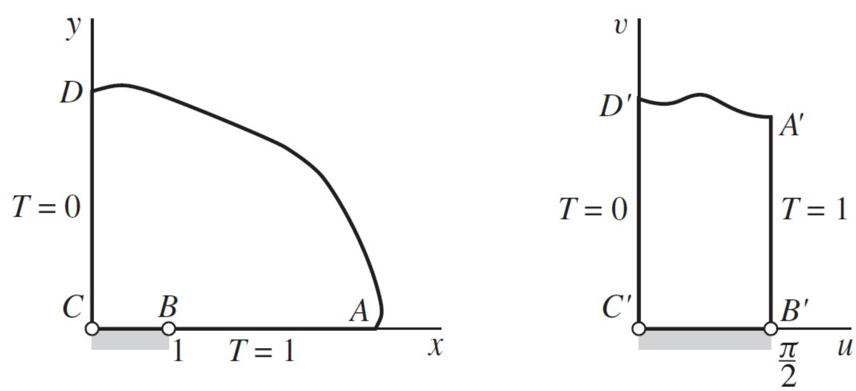


FIGURE 157

The boundary problem we are faced with is to find $T(x, y)$ satisfying Laplace's equation on the interior of the quadrant and satisfying the boundary conditions.

As in previous examples, we impose a condition of boundedness on $T(x, y)$. That is:

$$T_{xx}(x, y) + T_{yy}(x, y) = 0 \text{ for } x > 0,\ y > 0,$$

$$\begin{cases} T_y(x, 0) = 0 & \text{when } 0 < x < 1, \\ T(x, 0) = 1 & \text{when } x > 1, \end{cases}$$

$$T(0, y) = 0 \text{ for } y > 0,$$

and $|T(x, y)| < M$ for some positive constant M.

Note. Browsing Appendix 2, "Table of Transformations of Regions," we notice in Figure 10 that the transformation $z = \sin w$ transforms region in Figure 157 right to the first quadrant and that the boundary conditions transform as indicated in Figure 157.

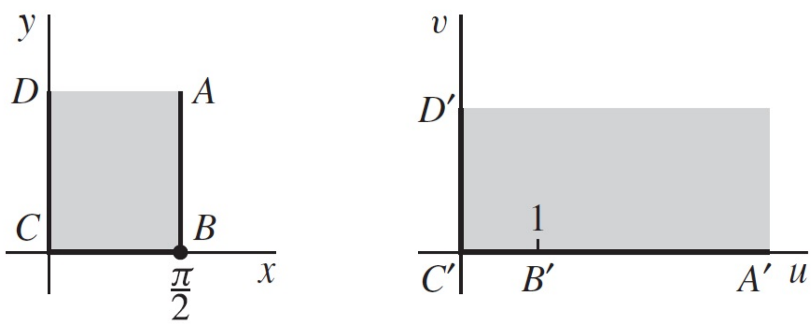


FIGURE 10
$w = \sin z$.

However, to apply Theorem 116.A and 117.A, we need to transform the first quadrant to the region on the left of Figure 10. Notice that $w = \sin z$ maps the region one to one and onto the first quadrant, so it has an inverse transformation. Interchanging w and z, the needed transformation is $z = \sin w$ (see Figure 10′). Notice that this is a conformal mapping, except when $z = 1$ and $w = \pi/2$ (when the derivative of $\sin w$ is 0). A solution to the boundary problem on the region in Figure 10′ right (see also Figure 157 right) is $T = (2/\pi)u = 2u/\pi$. This is harmonic, since it is the real part of analytic function $(2/\pi)w$.

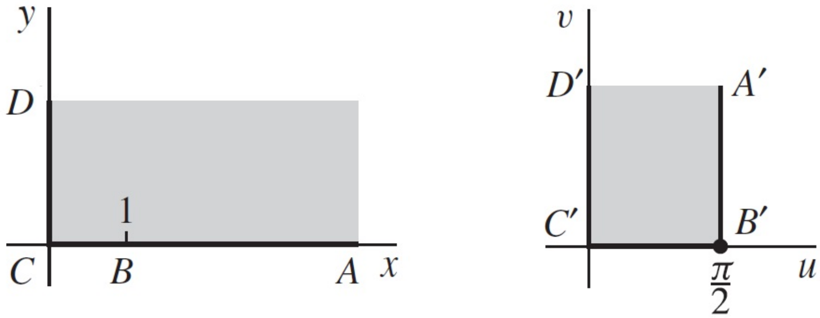


FIGURE 10′
$z = \sin w$.

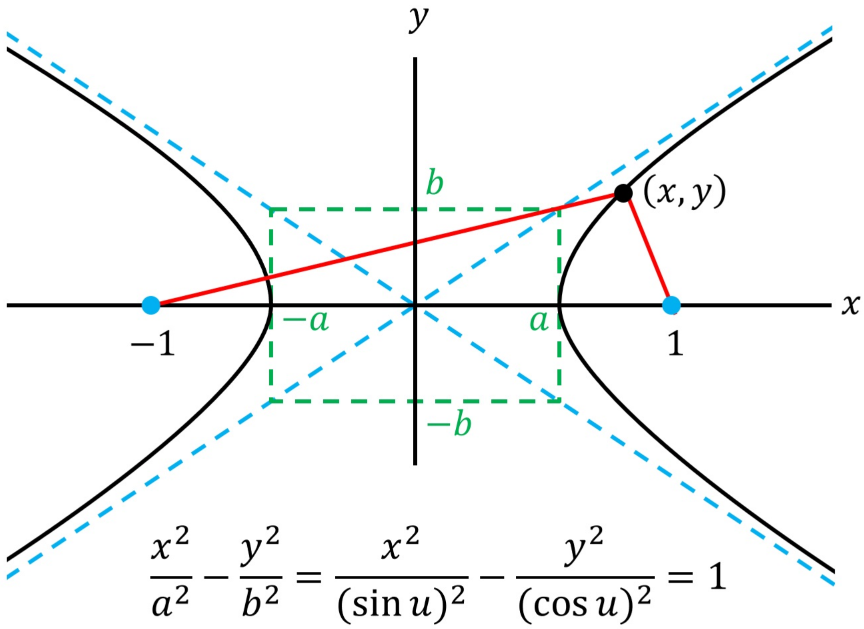
We now need w in terms of u and v. We have from Section 3.34. Trigonometric Functions (see Lemma 3.34.A) that

$$z = x + iy = \sin w = \sin u \cosh v + i \cos u \sinh v.$$

So we have $x = \sin u \cosh v$ and $y = \cos u \sinh v$. We now play some tricks to get u in terms of x and y. In the interior of the region in the w plane (Figure 10′ right) we have $0 < u < \pi/2$ so that both $\sin u$ and $\cos u$ are nonzero, so we have on the open region that

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = \frac{(\sin u \cosh v)^2}{\sin^2 u} - \frac{(\cos u \sinh v)^2}{\cos^2 u} = \cosh^2 v - \sinh^2 v = 1.$$

Recall that the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has its center at $(0,0)$, the axis of symmetry is $y = 0$, there vertices are $(\pm a, 0)$, and the foci are $(\pm c, 0)$ where $c^2 = a^2 + b^2$ (see below). With $a = \sin u$ and $b = \cos u$, this is the form of the above equation. Recall that a hyperbola is the set of all points in the xy plane such that the absolute value of the difference of the distances from a point on the hyperbola to the two foci is constant (see, for example, my online notes for Introduction to Modern Geometry [MATH 4157/5157] on Section 3.3. The Hyperbola and notice Theorem 3.3.A).



By considering the point $(a, 0)$ on the hyperbola, we see that the constant is $2a = 2\sin u$ in our case. Since $c^2 = a^2 + b^2 = \sin^2 u + \cos^2 u = 1$, then the foci are at $(\pm c, 0) = (\pm 1, 0)$ and the difference of distances from (x, y) (in the first quadrant) and the two foci satisfies

$$\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 2\sin u.$$

Since $x = \sin u \cosh v$ and $y = \cos u \sinh v$ form above, we have

$$\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}$$

$$\begin{aligned} &= \sqrt{(\sin u \cosh v + 1)^2 + (\cos u \sinh v)^2} - \sqrt{(\sin u \cosh v - 1)^2 + (\cos u \sinh v)^2} \\ &= \sqrt{\sin^2 u \cosh^2 v + 2\sin u \cosh v + 1 + \cos^2 u \sinh^2 v} \\ &\quad -\sqrt{\sin^2 u \cosh^2 v - 2\sin u \cosh v + 1 + \cos^2 u \sinh^2 v}. \end{aligned}$$

When $u = 0$ we have $\sin u = \sin 0 = 0$ and $\cos u = \cos 0 = 1$ and this reduces to

$$\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}$$

$$
\begin{aligned}
&= \sqrt{(0)^2 \cosh^2 v + 2(0)\cosh v + 1 + (1)\sinh^2 v} \\
&\quad - \sqrt{(0)^2 \cosh^2 v - 2(0)\cosh v + 1 + (1)^2 \sinh^2 v} \\
&= \sqrt{1 + \sinh^2 v} - \sqrt{1 + \sinh^2 v} = 0 = 2\sin 0.
\end{aligned}
$$

When $u = \pi/2$ we have $\sin u = \sin \pi/2 = 1$ and $\cos u = \cos \pi/2 = 0$ and this reduces to

$$
\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}
$$

$$
\begin{aligned}
&= \sqrt{(1)^2 \cosh^2 v + 2(1)\cosh v + 1 + (0)\sinh^2 v} \\
&\quad - \sqrt{(1)^2 \cosh^2 v - 2(1)\cosh v + 1 + (0)^2 \sinh^2 v} \\
&= \sqrt{\cosh^2 v + 2\cosh v + 1} - \sqrt{\cosh^2 v - 2\cosh v + 1} \\
&= |\cosh v + 1| - |\cosh v - 1| = (\cosh v + 1) - (\cosh v - 1) = 2 = \sin \pi/2
\end{aligned}
$$

where we drop the absolute value because $\cosh v > 0$ for all $v \in \mathbb{R}$. So the expression $\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2} = 2\sin u$ holds when $u = 0$ and $u = \pi/2$ also. Next, we have

$$
u = \arcsin\left(\frac{\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}}{2}\right)
$$

so that the solution to the original boundary problem is

$$
T(x,y) = \frac{2}{\pi} u = \frac{2}{\pi} \arcsin\left(\frac{\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}}{2}\right).
$$

Notice that the arcsine function returns values between 0 and $\pi/2$, so that the value of T is between 0 and 1, as expected. For $0 < x < 1$ and $y = 0$ (this represents the insulated part of the boundary of the first quadrant) we have that $T(x,0) = (2/\pi)\arcsin(((x+1) - (1-x))/2) = (2/\pi)\arcsin(x)$ (independent of y) so that the normal derivative is $T_y(x,0) = 0$ for $0 < x < 1$, as desired.

Note. For the isotherms, we set $T(x,y) = c_1$ where $0 < c_1 < 1$. This gives $u = \pi c_1/2$ so that the relationship above, $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$ becomes $\dfrac{x^2}{\sin^2(\pi c_1/2)} - \dfrac{y^2}{\cos^2(\pi c_1/2)} = 1$. These are hyperbolas with foci at $(\pm 1, 0)$, as described above, so that the isotherms are the portions of these hyperbolas that lie in the first quadrant.

Some isotherms are the following: