Chapter 2. Analytic Functions

Note. In this chapter we define complex-valued functions of a complex variable and explore their limits and derivatives.

Section 2.12. Functions of a Complex Variable

"Definition." A *function* f defined on $S \subset \mathbb{C}$ is a rule that assigns to each $z \in S$ a complex number w. The number w is called the *value* of f at z and is denoted $f(z)$. The set S is the *domain of definition* of f and we write $f : S \to \mathbb{C}$ (in which case, $\mathbb{C}$ is sometimes called the *codomain*). The set of values which f attains is the *range* of f, $\{z \in \mathbb{C} \mid w = f(z) \text{ for some } z \in \mathbb{C}\}$.

Note. At this level of class, we often develop the "definition" of function a bit more rigorously, based on a collection of ordered pairs of complex numbers such that no two distinct pairs have the same first entry. In fact, this is sometimes even done in our Precalculus 1 class (MATH 1710); see my online notes for this class on 2.1. Functions. But we will make do with Brown and Churchill's approach.

Note. With $w = u + iv$ as the value of f at $z = x + iy$, we have $u + iv = f(x + iy)$. Then each $u, v \in \mathbb{R}$ is a function of $x, y \in \mathbb{R}$, so we can write $u = u(x, y)$ and $v = v(x, y)$, so that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Similarly, in polar coordinates with $z = re^{i\theta}$ we can write $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.

Example 2.12.2. If $f(z) = z^2$ then

$$f(z) = f(x + iy) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

and so $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. In polar coordinates,

$$f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i(2\theta)} = r^2 \cos 2\theta + ir^2 \sin 2\theta,$$

and so $u(r, \theta) = r^2 \cos 2\theta$ and $v(r, \theta) = r^2 \sin 2\theta$.

Example 2.12.3. If $f(z) = |z|^2$ then $f(z) = f(x + iy) = (x^2 + y^2) + i(0)$ and so $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Similarly, $u(r, \theta) = r^2$ and $v(r, \theta) = 0$. This is a "real valued function of a complex variable" (which we are not particularly interested in; our interest lies with complex valued functions of a complex variable).

Definition. For $n \in \mathbb{N}$, if $a_0, a_1, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$, then the function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a *polynomial* of *degree* n. (Notice that Brown and Churchill also allow constant functions to be characterized as polynomials as well; this has advantages and disadvantages). A quotient $P(z)/Q(z)$ where P and Q are polynomials is a *rational function.*

Note. The domain of a polynomial is all of $\mathbb{C}$. The domain of rational function $P(z)/Q(z)$ is all of $\mathbb{C}$ except for the points where $Q(z) = 0$ (which are called *zeros* of polynomial Q).

Note. Brown and Churchill describes "multiple-valued functions" on page 37. In your humble instructor's opinion, this is a major weakness in the textbook. This can be dealt with in a more traditional way by introducing set-valued functions. **FUNCTIONS BY DEFINITION HAVE ONLY ONE OUTPUT!!!**

Example 2.12.4. We now consider the concept of a square root function. Recall that, in the real setting, the symbol "$\sqrt{x}$" denotes the positive square root of positive value x. So, for example, $\sqrt{9} = 3$; it is **false to write** $\boldsymbol{\sqrt{9} = \pm 3}$! If you "want" both the positive and negative square roots (which can certainly be the case in applied problems) then you have to "ask" for both the positive and negative square roots, so that it is **correct to write** $\boldsymbol{\pm\sqrt{9} = \pm 3}$. This is because the (real) square root function is a FUNCTION (with domain $[0, \infty)$). Brown and Churchill write (for the complex setting) $z^{1/2} = (re^{i\Theta})^{1/2} = \pm\sqrt{r}e^{i(\Theta/2)}$ where $r = |z|$ and $\Theta \in (-\pi, \pi]$ (so that Θ is the principal argument of z). But this is not actually a function (it is "2-valued"). Brown and Churchill accurately define the square root function $f(z)$ for any $z = re^{i\Theta} \neq 0$ (where Θ is the principal argument of z) as $f(z) = \sqrt{r}e^{i(\Theta/2)}$ where $\Theta \in (-\pi, \pi]$, and define $f(0) = 0$. In this way, the square root function has domain all of $\mathbb{C}$. Notice that the range of this function is $\{z \in \mathbb{C} \mid \arg(z) \in (-\pi/2, \pi/2]\} \cup \{0\}$ (the "right half plane," along with its boundary points which have nonnegative imaginary part).

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