Section 2.13. Mappings

Note. For a real valued function f of a real variable, we can graph the function in the Cartesian plane by plotting all points $(x, f(x))$ for x in the domain of f. We cannot perform the same feat for a complex valued function of a complex variable since this would require a "2-dimensional" domain and a "2-dimensional" range. We use the quotation marks because the term "dimension" is an aspect of a vector space. $\mathbb{C}$ is a 1-dimensional vector space with complex scalars, but our geometric interpretation of the complex plane as an "object" equivalent to $\mathbb{R}^2$ (we even associated vectors in $\mathbb{R}^2$ with complex numbers in Section 1.4) implies that we would need 4 (real) dimensions to graph $w = f(z)$. So we need to come up with something we can visualize. We often consider a particular set in the domain and graph its image in the range.

Definition. The *image* of a point x in the domain of function f is the point $w = f(z)$ in the range of f. For set S a subset of the domain of f, the set $T = \{w = f(z) \mid z \in S\}$ is the *image* of S. The *inverse image* of a point w in the range of function f is the set of all points z in the domain of f that have w as their image, $f^{-1}(\{w\}) = \{z \in \mathbb{C} \mid f(z) = w\}$. The *inverse image* of a set T in the range of function f is the set of values z that have their image in T, $f^{-1}(T) = \{z \in \mathbb{C} \mid f(z) \in T\}$.

Note. The existence of an inverse image is not to be confused with the existence of an inverse function. For example, consider $f(z) = z^4$. The image of $\{1, i, -1, -i\}$ is $f(\{1, i, -1, -i\}) = \{1\}$, and so the inverse image of $\{1\}$ is $f^{-1}(\{1\}) = \{1, i, -1, -i\}$. Notice that f is not a one to one function and so has no inverse function.

Note. Brown and Churchill use the terms "translation," "rotation," and "reflection" intuitively, though they can be given rigorous definitions.

Example 2.13.1. Consider $f(z) = z^2$. First, we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. So if we consider the hyperbola $x^2 - y^2 = c_1$ (where $c_1 > 0$, say) it is mapped to the vertical line in the w-plane of $u = x^2 - y^2 = c_1$ (and $v \in \mathbb{R}$), see Figure 17. If we consider the hyperbola $2xy = c_2$ (where $c_2 > 0$, say; this is a rotation of a hyperbola in standard form and has an axis with slope $m = 1$), it is mapped to the horizontal line $v = 2xy = c_2$ (and $u \in \mathbb{R}$), see Figure 17. Brown and Churchill give a more detailed argument about the parameterization of the vertical and horizontal lines (see pages 39 and 40).

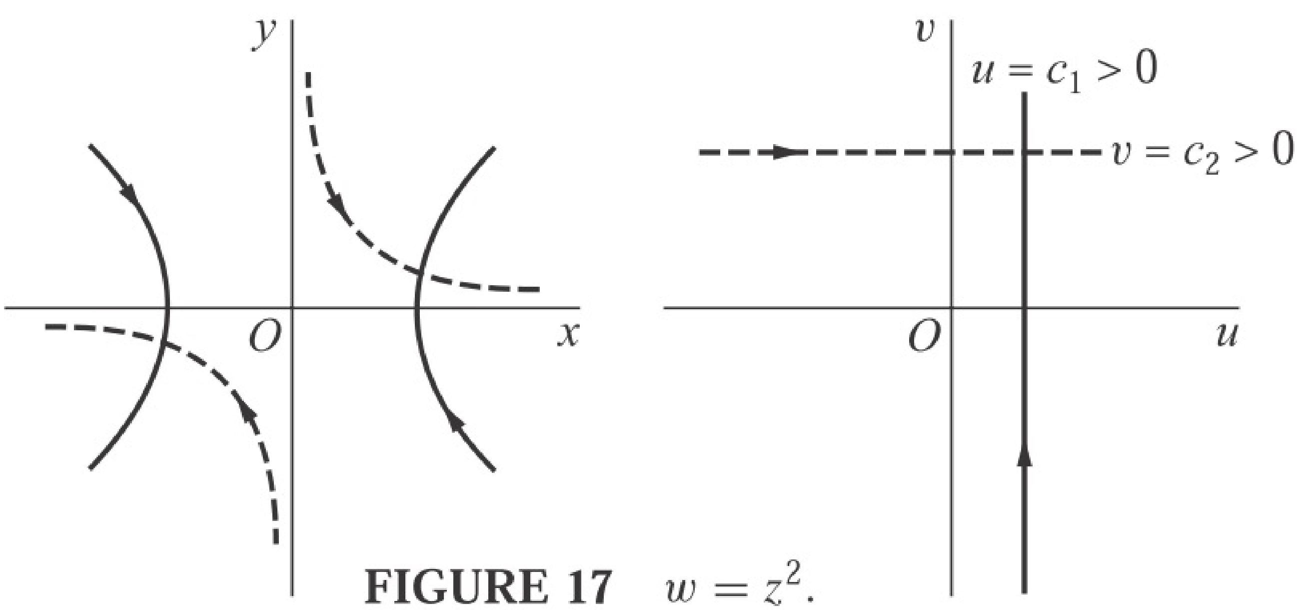

FIGURE 17 $w = z^2$.

Example 2.13.2. Consider $f(z) = z^2$ again. From Example 2.13.1, we consider the region consisting of hyperbolas of the form $2xy = c$ where $0 < c < 2$ (think of it as layers of hyperbolas). See Figure 18. This region is mapped to the horizontal strip $\{(u, v) \mid u \in \mathbb{R}, v \in (0, 2)\}$.

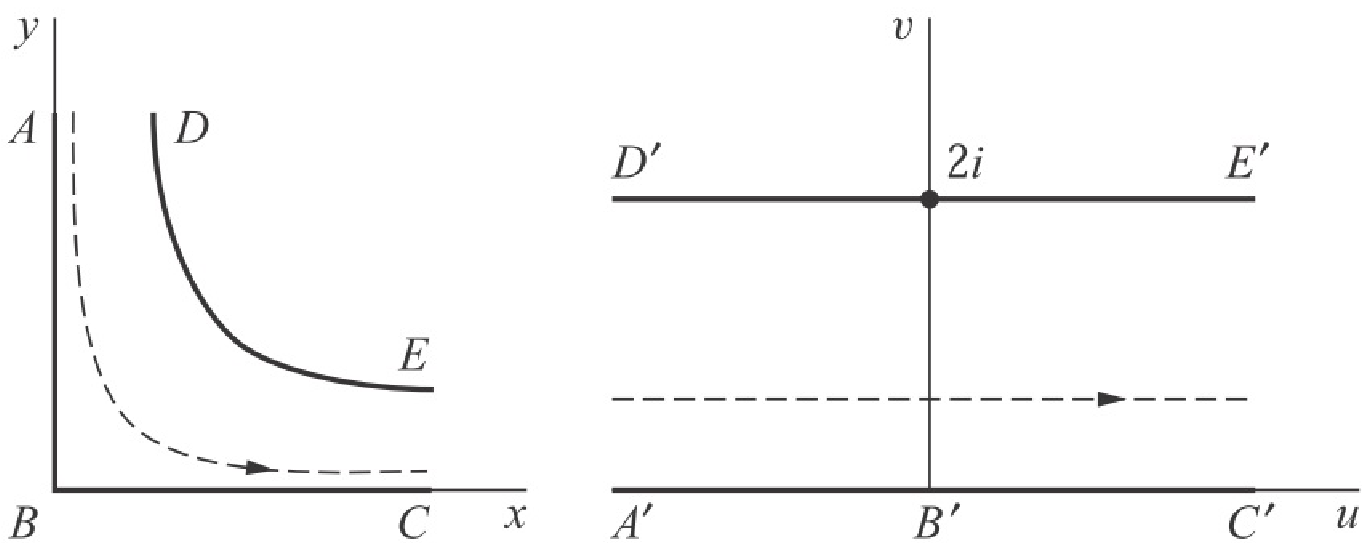

FIGURE 18 $w = z^2$.

Example 2.13.3. Consider $f(z) = z^2$ yet again, but this time we use polar coordinates and write $w = z^2 = r^2 e^{i2\theta}$. With $w = \rho e^{i\varphi}$ we have $\rho = r^2$ and $\varphi = 2\theta$. If we consider the circle $|z| = r_0$, then we see that it is mapped to the circle $|z| = \rho = r_0^2$. In fact, this is a 2 to 1 mapping since the upper half of $|z| = r_0^2$ (that is, $z = r_0 e^{i\theta}$ for $\theta \in [0, \pi)$) is mapped to $|z| = r_0^2$ and the lower half ($z = r_0 e^{i\theta}$ for $\theta \in [\pi, 2\pi)$) is also mapped to $|z| = r_0^2$. Of course, this is why function $f(z) = z^2$ does not have an inverse. Similarly, the upper half of the complex plane, $\mathrm{Im}(z) \geq 0$, is mapped by $f(z) = z^2$ to the whole complex plane.

Note 2.13.A. Similarly to the argument in Example 2.13.3, the function $f(z) = z^n$ for $n \in \mathbb{N}$ maps the circle $|z| = r_0$ to the circle $|w| = r_0^n$ in an n to 1 way. Also, $f(z) = z^n$ maps the sector $\{z \in \mathbb{C} \mid 0 \le \arg(z) < 2\pi/n\}$ to the whole complex plane.

Note. Figures 1, 2, and 3 of Appendix 2 give some mapping properties of $f(z) = z^2$.

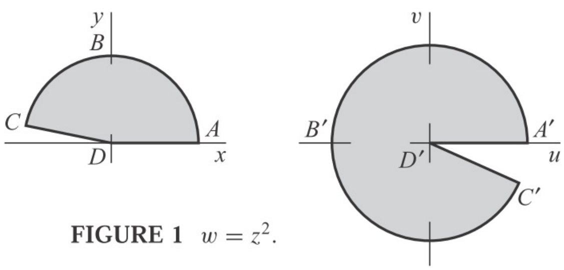

FIGURE 1 $w = z^2$.

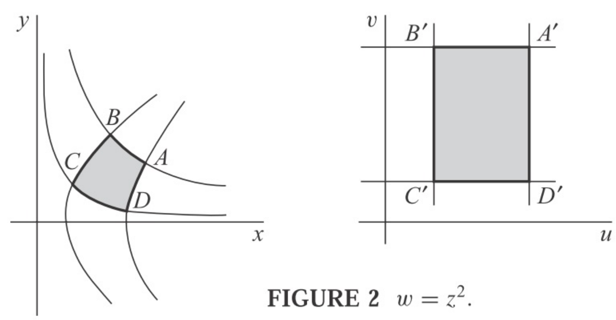

FIGURE 2 $w = z^2$.

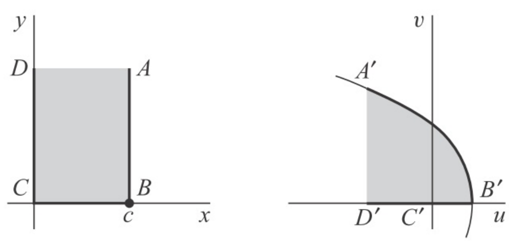

FIGURE 3 $w = z^2$; $A'B'$ on parabola $v^2 = -4c^2(u - c^2)$.