Section 2.14. Mappings by the Exponential Function

Note. In this section, we (finally) define the complex exponential function. We then explore some mapping properties implied by the definition.

Definition. Define the *exponential function* e^z the as $e^z = e^{x+iy} = e^x e^{iy}$.

Note. Of course e^x for $x \in \mathbb{R}$ is defined and e^{iy} is defined by Euler's formula $e^{iy} = \cos y + i \sin y$. We have $|e^z| = |e^x e^{iy}| = |e^x|(1) = e^x$. Since $e^x > 0$ for all $x \in \mathbb{R}$, then $e^z \neq 0$ for all $z \in \mathbb{C}$. In graduate-level Complex Analysis 1 (MATH 5510), we take a different approach to defining e^z and define it in terms of a series (see my online notes on III.1. Power Series).

Example 2.14.1. In polar coordinates $w = \rho e^{i\varphi}$ we have from $e^z = e^{x+iy} = e^x e^{iy}$ that $\rho = e^x$ and $\varphi = y$. So a point $z = c_1 + iy$ on the vertical line $x = c_1$ in the z-plane is mapped by $f(z) = e^z$ to the point $w = \rho e^{i\varphi} = e^{c_1} e^{iy}$ in the w-plane which lies on the circle $\rho = e^{c_1}$ in the w-plane and as y varies (as the point $z = c_1 + iy$ moves along the vertical line $x = c_1$ in the z-plane) the image w moves around the circle $\rho = e^{c_1}$ as shown in Figure 20. Notice that geometrically the line is mapped around the circle an infinite number of times. The horizontal line $y = c_2$ in the z-plane is mapped by $f(z) = e^z$ onto the ray $\varphi = c_2$ and $\rho > 0$. See Figure 20. This is a one to one mapping.

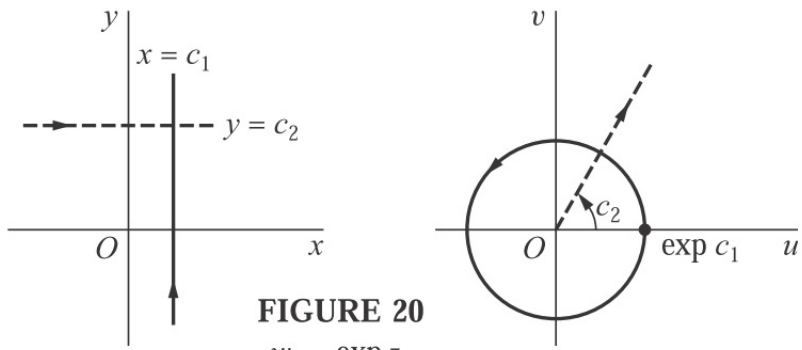

FIGURE 20
$w = \exp z$.

Example 2.14.2. Consider the rectangle in the z-plane $\{x + iy \in \mathbb{C} \mid a \le x \le b, c \le y \le d\}$. Then $f(z) = e^z$ maps this to $\{\rho e^{i\varphi} \in \mathbb{C} \mid e^a \le \rho \le e^b, c \le \varphi \le d\}$. See Figure 21.

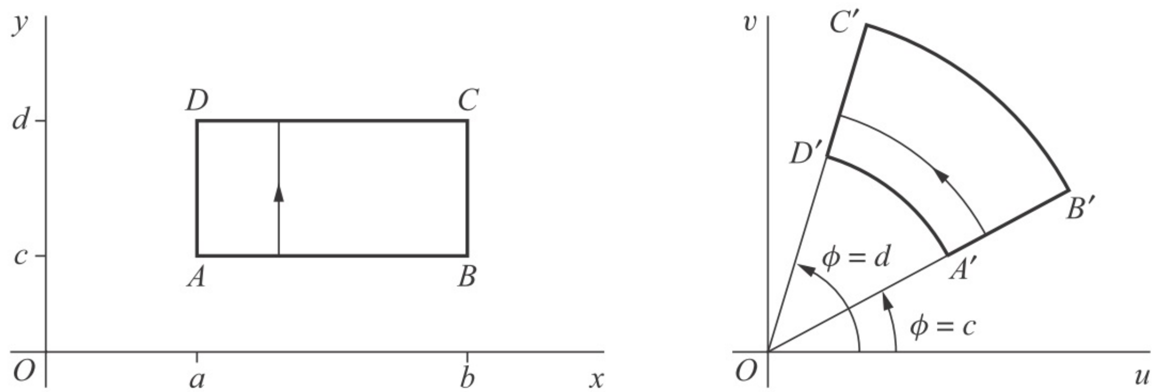

FIGURE 21 $w = \exp z$.

Example 2.14.3. Since horizontal lines are mapped to rays, the strip $0 \le y \le \pi$ in the z-plane is mapped to the upper half-plane $v \ge 0$ of the w-plane (except for $0 \in w$). Also, any horizontal strip of height 2π (and infinite width) is mapped to the entire w-plane except 0. Therefore $f(z) = e^z$ maps $\mathbb{C}$ to $\mathbb{C}$ in an infinite to

1 way (in fact, $f(z) = e^z$ is a periodic function of period $2\pi i$). This has intense implications in defining an inverse of e^z.

Note. In Figures 6, 7, and 8 of Appendix 2, these mappings are further illustrated.

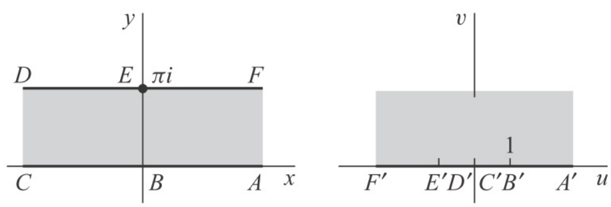

FIGURE 6 $w = \exp z$.

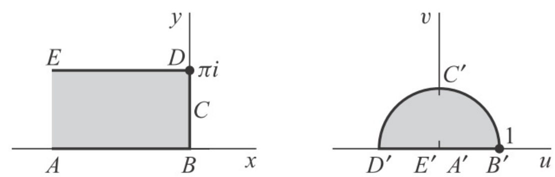

FIGURE 7 $w = \exp z$.

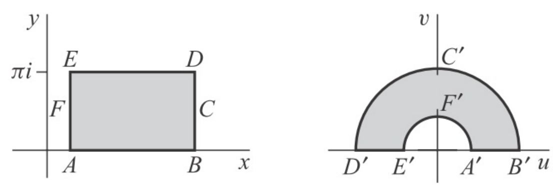

FIGURE 8 $w = \exp z$.

Revised: 2/21/2024