Section 2.15. Limits

Note. In this section we give an ε/δ definition of limit like in Calculus 1 (see my online notes on 2.3. The Precise Definition of Limit). Basically, we just replace absolute value on $\mathbb{R}$ with modulus on $\mathbb{C}$.

Definition. Let f be a function defined at all points z in some deleted neighborhood of z_0. If there is $w_0 \in \mathbb{C}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } |f(z) - w_0| < \varepsilon,$$

then the *limit* of f as z approaches z_0 is w_0, denoted $\lim_{z \to z_0} f(z) = w_0$.

Note. As in the real setting, $\lim_{z \to z_0} f(z) = w_0$ means that $f(z)$ and w_0 can be made arbitrarily close together by making z sufficiently close to z_0. (As always, the $\varepsilon > 0$ comes *first* and *then* the $\delta > 0$ is determined based on the given $\varepsilon > 0$.) See Figure 23.

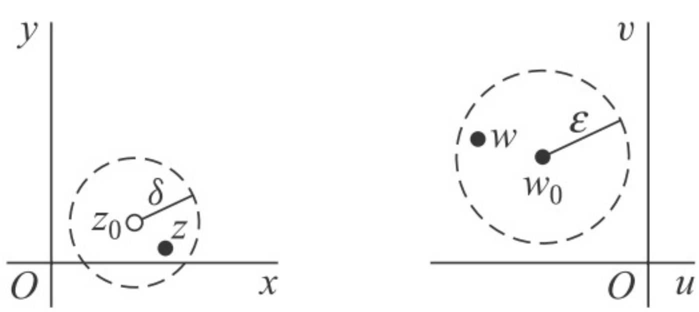

FIGURE 23

Lemma 2.15.A. Let f be a function defined at all points z in some deleted neighborhood of z_0. If $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} f(z) = w_1$, then $w_0 = w_1$.

Note. Brown and Churchill next define $\lim_{z\to z_0} f(z)$ where z_0 is a boundary point of the domain of f. This covers the idea of limit in the most general case (notice that this surpasses the definition of limit in Calculus 1, but is consistent with the definition you see in senior level Analysis 1 [MATH 4217/5217]; see my online notes on 4-1. Limits and Continuity).

Definition. Let f be a function and let z_0 be a boundary point of the domain of f. If there is $w_0 \in \mathbb{C}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ and } z \text{ is in the domain of } f \text{ implies } |f(z) - w_0| < \varepsilon,$$

then the *limit* of f as z approaches z_0 is w_0, denoted $\lim_{z\to z_0} f(z) = w_0$.

Example 2.15.1. Let $f(z) = i\overline{z}/2$. Notice that the domain of definition is all of $\mathbb{C}$. Let $z_0 = 1$ and $\varepsilon > 0$. We choose $\delta = 2\varepsilon > 0$. Then $|z - z_0| < \delta = 2\varepsilon$ implies $|z - 1| < 2\varepsilon$ or $|z - 1|/2 < \varepsilon$ or

$$|\overline{z-1}|/2 = |\overline{z} - 1|/2 = |i|\,|\overline{z} - 1|/2 = |i\overline{z} - i|/2 = |i\overline{z}/2 - i/2| = |f(z) - i/2| < \varepsilon.$$

That is, $|f(z) - w_0| < \varepsilon$ where $w_0 = i/2$. So $\lim_{z\to z_0} f(z) = \lim_{z\to 1} i\overline{z}/2 = w_0 = i/2$.

Example 2.15.2. Define $f(z) = z/\overline{z}$. Notice the domain of definition is $\mathbb{C} \setminus \{0\}$, so f is defined at all points x in the deleted neighborhood $\mathbb{C} \setminus \{0\}$ of 0. We now show $\lim_{z \to z_0} f(z) = \lim_{z \to 0} z/\overline{z}$ does not exist using the definition of limit. Consider $\varepsilon = 1/2$. Then for *any* $\delta > 0$, we consider first $z = \delta/2$. We have $0 < |z - z_0| = |\delta/2 - 0| = \delta/2 < \delta$ and $|f(z) - w_0| = |(\delta/2)/(\overline{\delta/2}) - w_0| = |1 - w_0|$. Second, consider $z = \delta i/2$. We have $0 < |z - z_0| = |\delta i/2 - 0| = \delta/2 < \delta$ and $|f(z) - w_0| = |(\delta i/2)/(\overline{\delta i/2}) - w_0| = |-1 - w_0|$. *If* the limit exists, then for *some* $\delta > 0$, we must have both $|1 - w_0| < \varepsilon = 1/2$ and $|-1 - w_0| < \varepsilon = 1/2$. But there is no w_0 which is both within distance 1/2 of 1 and within distance 1/2 of -1. So for $\varepsilon = 1/2 > 0$, there is no corresponding $\delta > 0$ satisfying the definition of limit for this $f(z)$ and z_0. Therefore, the limit does not exist.

Note 2.15.A. Brown and Churchill explain Example 2.15.2 slightly differently. They argue that for any z on the real axis $f(z) = 1$, and for any z on the imaginary axis $f(z) = -1$. So the limit as z approaches 0 along the real axis is 1 and the limit as z approaches 0 along the imaginary axis is -1. Since the limits along these two paths are different, then the limit does not exist. Our argument uses the definition of limit and Brown and Churchill are using (without stating it) the Two-Path Test for Nonexistence of a Limit (see page 10 of my online notes for Calculus 3 at 14.2. Limits and Continuity in Higher Dimensions where the Two-Path Test is stated in the setting of a function of two real variables).

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