# **Section 2.16.** Theorems on Limits

**Note.** In this section we state and prove two useful theorems on limits. First we recall Thomas' definition of limit in the setting of a function of two real variables. This definition is also in my online notes for Calculus 3; see page 1 of 14.2. Limits and Continuity in Higher Dimensions:

**Thomas' Definition of Limit.** Let function $f(x, y)$ be defined in a deleted neighborhood of $(x_0, y_0)$. We say that $f(x, y)$ approaches the *limit* $L$ as $(x, y)$ approaches $(x_0, y_0)$, denoted $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \text{ implies } |f(x, y) - L| < \epsilon.$$

**Theorem 2.16.1.** Suppose that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

**Note.** The proof of the second theorem follows easily from the first if we use some of the results from Calculus 3. We take the following as given. It is Theorem 1 in my online notes on 14.2. Limits and Continuity in Higher Dimensions:

**Thomas' Theorem 1. Properties of Limits of Functions of Two Variables.** The following rules hold if $L$, $M$, and $k$ are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

**1.** *Sum Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$

**2.** *Difference Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$

**3.** *Constant Multiple Rule*: $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number $k$)

**4.** *Product Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)g(x,y)) = LM$

**5.** *Quotient Rule*: $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$

**6.** *Power Rule*: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$, $n$ a positive integer

**7.** *Root Rule*: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, $n$ a positive integer and if $n$ is even, we assume $L \geq 0$.

**Theorem 2.16.2.** Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Then

$$\begin{aligned}
\lim_{z\to z_0} (f(z) + F(z)) &= w_0 + W_0 \\
\lim_{z\to z_0} f(z)F(z) &= w_0 W_0, \text{ and} \\
\lim_{z\to z_0} f(z)/F(z) &= w_0/W_0 \text{ if } W_0 \neq 0
\end{aligned}$$

**Note 2.16.A.** From the second claim of Theorem 2.16.2, if $f(z) = c$ for some constant $c \in \mathbb{C}$, then we have $\lim_{z \to z_0} cF(z) = cw_0$. This combines with the first claim of Theorem 2.16.2 to show that limits behave in a linear way. Namely, for any constant $c_1, c_2 \in \mathbb{C}$ we have $\lim_{z \to z_0}(c_1 f(z) + c_2 F(z)) = c_1 w_0 + c_2 W_0$.

**Lemma 2.16.A.** For any $z_0, c \in \mathbb{C}$, we have $\lim_{z \to z_0} c = c$ and $\lim_{z \to z_0} z = z_0$.

**Lemma 2.16.B.** For any $z_0 \in \mathbb{C}$ and $n \in \mathbb{N}$, we have $\lim_{z \to z_0} z^n = z_0^n$.

**Corollary 2.16.A.** Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n$. Then $\lim_{z \to z_0} P(z) = P(z_0)$.

**Corollary 2.16.B.** Let $R(z) = P_1(z)/P_2(z)$ be a rational function; that is, $R$ is the quotient of polynomials $P_1$ and $P_2$. Then $\lim_{z \to z_0} R(z) = R(z_0)$, provided $P_2(z_0) \neq 0$.

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