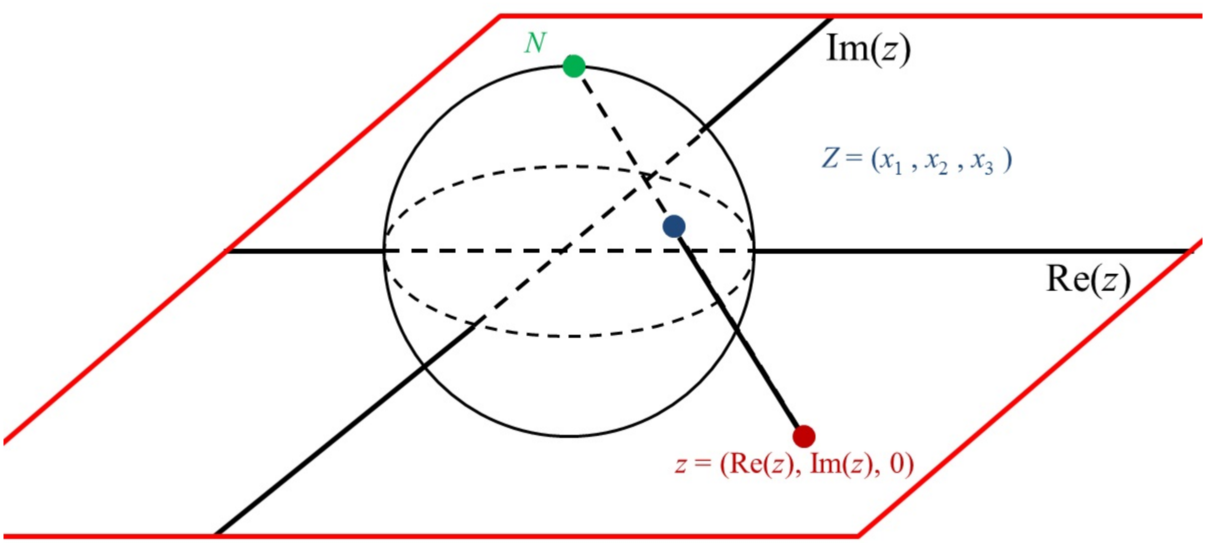
# **Section 2.17.** Limits Involving the Point at Infinity

**Note.** In this section, we introduce the symbol $\infty$ and rigorously define limits of $f(z)$ as $z$ approaches $\infty$ and limits of $f(z)$ which are $\infty$.

**Note.** Brown and Churchill introduce a sphere of radius 1 centered at the origin of the complex plane. They define the point $N$ as the point on the sphere farthest from the plane and "above" the plane (corresponding, in a sense, to the "north pole" of the sphere). They then map each point $P$ on the sphere (other than $N$) onto the plane by projecting the point $P$ onto the plane with a straight line through $P$ and $N$. This is called the *stereographic projection* and the sphere is called the *Riemann sphere*. The point $N$ itself is then associated with the symbol $\infty$. In this way, we have a one to one and onto mapping (i.e., a bijection) from the Riemann sphere to $\mathbb{C} \cup \{\infty\}$ (which is called the *extended complex plane*).



**Note.** In Introduction to Topology (MATH 4357/5357), you will encounter the extended complex plane as a "one-point compactification" of the complex plane; see my online notes for Introduction to Topology at 29. Local Compactness (see Example 4). We also address the extended complex plane as a metric space in our graduate-level Complex Analysis 1 class (MATH 5510); see my notes for this class at I.6. The Extended Plane and its Spherical Representation.

**Definition.** In the extended complex plane, an *$\varepsilon$-neighborhood of $\infty$* is the set $\{z \in \mathbb{C} \mid 1/|z| < \varepsilon\}$. An open set containing an $\varepsilon$-neighborhood of $\infty$ for some $\varepsilon > 0$ is a *neighborhood of $\infty$*.

**Definition.** Let $f$ be a function defined at all points $z$ of some neighborhood of $\infty$. If there is $w_0 \in \mathbb{C}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$1/|z| < \delta \text{ implies } |f(z) - w_0| < \varepsilon,$$

then the *limit as $z$ approaches $\infty$* of $f$ is $w_0$, denoted $\lim_{z\to\infty} f(z) = w_0$.

**Definition.** Let $f$ by a function defined and nonzero at all points $z$ in some deleted neighborhood of $z_0$. If for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } 1/|f(z)| < \varepsilon,$$

then the *limit of $f$ as $z$ approaches $z_0$ is $\infty$* , denoted $\lim_{z\to z_0} f(z) = \infty$.

**Definition.** Let $f$ be a function defined and nonzero at all points $z$ of some neighborhood of $\infty$. If for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$1/|z| < \delta \text{ implies } 1/|f(z)| < \varepsilon$$

then the *limit as $z$ approaches $\infty$* of $f$ is $\infty$, denoted $\lim_{z\to\infty} f(z) = \infty$.

**Theorem 2.17.1.** If $z_0, w_0 \in \mathbb{C}$ then

$$\begin{aligned} \lim_{z\to z_0} f(z) = \infty & \text{ if and only if } \lim_{z\to z_0} 1/f(z) = 0 \\ \lim_{z\to\infty} f(z) = w_0 & \text{ if and only if } \lim_{z\to 0} f(1/z) = w_0, \text{ and} \\ \lim_{z\to\infty} f(z) = \infty & \text{ if and only if } \lim_{z\to 0} 1/f(1/z) = 0. \end{aligned}$$

**Example 2.17.A.** We now establish the following limits.

**(a)** $\displaystyle\lim_{z\to -1} \frac{iz+3}{z+1} = \infty.$

**Solution.** Notice that the domain of $(iz+3)/(z+1)$ is $\mathbb{C} \setminus \{-1\}$, so it is defined on any deleted neighborhood of $z_0 = -1$. Also, $(iz+3)/(z+1) = 0$ only for $z = -3/i = 3i$, so it is nonzero on a deleted $\varepsilon$ neighborhood of $-1$ where $0 < \varepsilon < |-1-3i| = \sqrt{10}$. We let $f(z) = \dfrac{iz+3}{z+1}$ and consider

$$\lim_{z\to -1} \frac{1}{f(z)} = \lim_{z\to -1} \frac{z+1}{iz+3} = \frac{(-1)+1}{i(-1)+3} = \frac{0}{3-i} = 0$$

where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (1st claim), $\displaystyle\lim_{z\to -1} f(z) = \lim_{z\to -1} \frac{iz+3}{z+1} = \infty.$ $\square$

**(b)** $\displaystyle\lim_{z\to\infty} \frac{2z+i}{z+1} = 2.$

**Solution.** Notice that the domain of $(2z+i)/(z+1)$ is $\mathbb{C} \setminus \{-1\}$, so it is defined on a neighborhood of $\infty$. We let $f(z) = \dfrac{2z+i}{z+1}$ and consider

$$\lim_{z\to 0} f(1/z) = \lim_{z\to 0} \frac{2(1/z)+1}{(1/z)+1} = \lim_{z\to 0} \frac{2(1/z)+1}{(1/z)+1}\frac{z}{z} = \lim_{z\to 0} \frac{2+iz}{1+z} = \frac{2+i(0)}{1+(0)} = \frac{2}{1} = 2$$

where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (2nd claim), $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \dfrac{2z+i}{z+1} = 2$. □

**(c)** $\lim_{z\to\infty} \dfrac{2z^3-1}{z^2+1} = \infty$.

**Solution.** Notice that the domain of $(2z^3-1)/(z^2+1)$ is $\mathbb{C}\setminus\{-i,i\}$, so it is defined on a neighborhood of $\infty$. Also, $(2z^3-1)/(z^2+1) = 0$ only for $z$ equal to the three cube roots of $1/2$, so it is nonzero on a neighborhood of $\infty$. For example, we can take $\{z \in \mathbb{C} \mid 1/|z| < 1\}$, and $(2z^3-1)/(z^2+1)$ is defined and nonzero on this neighborhood of $\infty$. We let $f(z) = \dfrac{2z^3-1}{z^2+1}$ and consider

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{(1/z)^2+1}{2(1/z)^3-1} = \lim_{z\to 0} \frac{(1/z)^2+1}{2(1/z)^3-1}\frac{z^3}{z^3} = \lim_{z\to 0} \frac{z+z^3}{2-z^3}$$

$$= \lim_{z\to 0} \frac{(0)+(0)^3}{2-(0)^3} = \frac{0}{2} = 0$$

where we have evaluated the limit using Corollary 2.16.B. So by Theorem 2.17.1 (3rd claim), $\lim_{z\to\infty} f(z) = \infty$. □

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