Section 2.18. Continuity

Note. In this section, we define the continuity of a function similar to the way it is done in Calculus 1. Remember that we have a slightly different definition of limit for boundary points of the domain of a function, so this will have an implication for continuity at such points. This is also the case in Calculus 1 where continuity at the endpoints of the domain are defined separately; see my online Calculus 1 notes on 2.5. Continuity.

Definition. For z_0 an interior point or a boundary point of the domain of function f, f is *continuous at point* z_0 if

(1) $\lim_{z\to z_0} f(z)$ exists,

(2) $f(z_0)$ exists, and

(3) $\lim_{z\to z_0} f(z) = f(z_0)$.

Function f is *continuous on region* R (an open connected set along with none, some, or all of its boundary points) if f is continuous at each point of R.

Note. We first have two easy results which follow from limit properties as given in Section 2.16.

Corollary 2.18.A. If f and g are continuous at point z_0, an interior or boundary point of the domain of f and g, then $f + g$, $f - g$, and fg are continuous at z_0. If in addition $g(z_0) \neq 0$ then f/g is continuous at z_0.

Proof. The proof follows from the corresponding result for limits, Theorem 2.16.2. □

Corollary 2.18.B. For a polynomial function p, p is continuous in the entire complex plane.

Proof. The proof follows from Corollary 2.16.A. □

Note. We now have two results, the proofs of which depend on the ε/δ definition of limit in part (1) of the definition of continuity.

Theorem 2.18.1. If f is continuous at z_0 and g is continuous at $f(z_0)$ (so z_0 is an interior or boundary point of the domain of f and $f(z_0)$ is an interior or boundary point of the domain of g), then $(g \circ f)(z) = g(f(z))$ is continuous at z_0.

Theorem 2.18.2. If f is continuous at z_0 (an interior or boundary point of the domain of f) and $f(z_0) \neq 0$ then $f(z) \neq 0$ throughout some neighborhood of z_0.

Note. The idea of continuity can be extended to functions from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}^n$ (or $\mathbb{C}^n$); in fact, the idea can be extended to any mappings between metric spaces...even between topological spaces. By the Heine-Borel Theorem, a "compact set" in $\mathbb{R}^n$ or $\mathbb{C}^n$ is a closed and bounded set. A continuous function maps compact sets to compact sets (see my online notes for 3.1. Topology of the Real Numbers and 4.1. Limits and Continuity. from Analysis 1 [MATH 4127/5127]) . These deeper analysis ideas can be used to prove the following.

Theorem 2.18.3. If a function f is continuous throughout a region R that is both closed and bounded, then there exists $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in R$, where equality holds for at least one z.

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