Section 2.19. Derivatives

Note. Our definition of derivative is the same as in Calculus 1 (although our definition of limit is arguably more restrictive).

Definition. Let f have a domain which contains a neighborhood of z_0, $\{z \mid |z - z_0| < \varepsilon\}$ for some $\varepsilon > 0$. The *derivative of f at z_0* is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided the limit exists. Function f is *differentiable at z_0* when $f'(z_0)$ exists. We can define the derivate of f as a function of the z-values where the derivative is defined and denote this function as $f'(z)$, the *derivative of f*. We often let $w = f(z)$ and denote $f'(z) = dw/dz$.

Example 2.19.1. Let $f(z) = z^2$. Then

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} (2z + \Delta z) \text{ since “} \lim_{\Delta z \to 0} \text{” implies } \Delta z \neq 0 \\
&= 2z + (0) = 2z \text{ by Corollary 2.16.A, since } 2z + \Delta z \text{ is a polynomial in } \Delta z.
\end{aligned}$$

Example 2.19.2. Let $f(z) = \overline{z}$, then

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}.$$

As in Note 2.15.A, we can apply the Two-Path Test for Nonexistence of a Limit and consider the limit $\Delta z = \Delta x + i\Delta y \to 0$ along the real axis, $\Delta z = \Delta x + i(0) = \Delta x \to 0$, and along the imaginary axis, $\Delta z = 0 + i\Delta y = i\Delta y \to 0$, as in Figure 29.

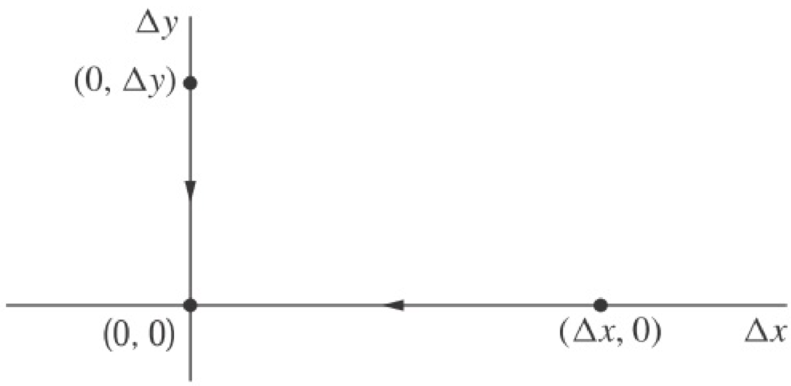

FIGURE 29

Along these paths we have

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

and

$$\lim_{i\Delta y \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{i\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{i\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = \lim_{i\Delta y \to 0} -1 = -1.$$

Since the two paths give different limits, then the limit cannot exist. That is, $f(z) = \overline{z}$ is not differentiable at any $z \in \mathbb{C}$. This approach to studying differentiability fore shadows our approach in Section 2.21 ("The Cauchy-Riemann Equations").

Example 2.19.3. Let $f(z) = |z|^2$, then

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z}) - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z}) - |z|^2}{\Delta z}$$

$$= \lim \Delta z \to 0 \frac{(z+\Delta z)(\overline{z}+\overline{\Delta z}) - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\overline{z} + \Delta z\overline{z} + z\overline{\Delta z} + \Delta z\overline{\Delta z} - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z|^2 + \Delta z(\overline{z}+\overline{\Delta z}) + z\overline{\Delta z} - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \left(\overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \right).$$

As in Example 2.19.2, using the Two-Path Test,

$$\lim_{\Delta x \to 0} \left(\overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \right) = \lim_{\Delta x \to 0} \left(\overline{z} + \overline{\Delta x} + z\frac{\overline{\Delta x}}{\Delta x} \right) = \lim_{\Delta x \to 0} \left(\overline{z} + \Delta x + z\frac{\Delta x}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} (\overline{z} + \Delta x + z) = \overline{z} + z,$$

and

$$\lim_{i\Delta y \to 0} \left(\overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \right) = \lim_{i\Delta y \to 0} \left(\overline{z} + \overline{i\Delta y} + z\frac{\overline{i\Delta y}}{i\Delta y} \right) = \lim_{i\Delta y \to 0} \left(\overline{z} - i\Delta y + z\frac{-i\Delta y}{i\Delta y} \right)$$

$$= \lim_{i\Delta y \to 0} (\overline{z} - i\Delta y + z(-1)) = \overline{z} - (0) - z = \overline{z} - z,$$

Now these two limits, $\overline{z} + z$ and $\overline{z} - z$, are not equal unless $z = 0$. So if $z \neq 0$ then the limits are different and $f(z) = |z|^2$ is not differentiable. Next, we must check $z = 0$:

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = \overline{0} = 0.$$

So $f'(0) = 0$ and f *is* differentiable at $z = 0$.

Note. Example 2.19.3 shows that a function f can be continuous at every point ($f(z) = |z|^2$ or $f(x+iy) = x^2 + y^2$ is continuous at all $z = x + iy$) but may not be differentiable. However we have, as in Calculus 1 (see my online Calculus 1 notes on Section 3.2. The Derivative as a Function; notice Theorem 3.1), the following.

Theorem 2.19.A. Differentiable implies Continuous. If f is differentiable at point z_0 then f is continuous at z_0.

Revised: 3/24/2024