# **Section 2.20.** Differentiation Formulas

**Note.** Since the definition of derivative in the complex setting is the same as the definition in the real setting (though, technically, the definition of "limit" is more involved in the complex setting) most of the properties of derivatives from Calculus 1 carry over to our current setting.

**Theorem 2.20.A.** Let $c \in \mathbb{C}$ and let $f$ and $g$ be functions where derivatives exist at a point $z \in \mathbb{C}$. Then:

$$\frac{d}{dz}[c] = 0,\ \frac{d}{dz}[z] = 1,\ \frac{d}{dz}[cf(z)] = c\frac{d}{dz}[f],$$
$$\text{and } \frac{d}{dz}[f(z) + g(z)] = \frac{d}{dz}[f(z)] + \frac{d}{dz}[g(z)].$$

**Note.** The last two properties of Theorem 2.20.A together imply that the derivative of a linear combination of differentiable functions is the linear combination of the derivatives:

$$\frac{d}{dz}[af(z) + bg(z)] = a\frac{d}{dz}[f(z)] + b\frac{d}{dz}[g(z)]$$

where $a, b \in \mathbb{C}$. For this reason, differentiation is an example of a "linear operator."

**Note.** The familiar product and quotient rules hold. Again, we often denote $\frac{d}{dz}[f(z)] = f'(z)$.

**Theorem 2.20.B.** If $f$ and $g$ are functions whose derivative exists at a point $z$ then

$$\text{Product Rule: } \frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \text{ and}$$

$$\text{Quotient Rule: } \frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} \text{ if } g(z) \neq 0.$$

**Corollary 2.20.A.** If $n \in \mathbb{N}$ then $\dfrac{d}{dz}[z^n] = nz^{n-1}$.

**Corollary 2.20.B.** If $n \in \mathbb{Z}$ then $\dfrac{d}{dz}[z^n] = nz^{n-1}$.

**Proof.** Corollary 2.20.A establishes the claim for $n \in \mathbb{N}$. For $n = 0$, Theorem 2.20.A establishes the claim (since $z^0 = 1$, a constant). For $n \in \mathbb{Z}$ with $n < 0$, we have $-n > 0$ and so by the Quotient Rule and Corollary 2.20.A:

$$\frac{d}{dz}[z^n] = \frac{d}{dz}\left[\frac{1}{z^{-n}}\right] = \frac{[0](z^{-n}) - (1)[-nz^{-n-1}]}{(z^{-n})^2} = \frac{nz^{-n-1}}{z^{-2n}} = nz^{n-1}.$$

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**Theorem 2.20.C. The Chain Rule.**

Suppose that $f$ has a derivative at $z_0$ and that $g$ has a derivative at $f(z_0)$. Then the composition function $F(z) = (g \circ f)(z) = g(f(z))$ has a derivative at $z_0$ and $F'(z_0) = g'(f(z_0))f'(z_0)$. In differential notation with $w = f(z)$ and $W = g(w)$ (so that $W = F(z)$) we have $\dfrac{dW}{dz} = \dfrac{dW}{dw}\dfrac{dw}{dz}$.

**Note.** I often use a notation for derivatives which involves putting the differentiated quantity in square brackets (motivated by the differentiation operator notation "$\frac{d}{dz}[\ ]$"). This is explained in: R. Gardner, A Useful Notation for Rules of Differentiation, *The College Journal of Mathematics*, **24**(4) (1993) 351-352 (also reprinted in *The Calculus Collection: A Resource for AP and Beyond*, pages 257-58, edited by C. Diefenderfer and R. Nelsen, The Mathematical Association of America [2010], and available online here). This allows us to draw diagrams of the Product and Quotient Rules as follows:

$$\frac{d}{dz}[(\ )(\ )] = [\ ](\ ) + (\ )[\ ] \text{ and } \frac{d}{dz}\left[\frac{(\ )}{(\ )}\right] = \frac{[\ ](\ ) - (\ )[\ ]}{(\ )^2}.$$

The special case of the Chain Rule, called the "Power Rule" in Calculus 1, can be written as:

$$\frac{d}{dz}[(\ )^n] = n(\ )^{n-1}[\ ].$$

This notation converts complicated derivative computations into simple fill in the blank problems.

**Example.** If $f(x) = (5x^4 - x)^3(9x^2 - 1)^5/(5x^5 + x)$ then

$$f'(x) = \frac{[[3(5x^4 - x)^2[20x^3 - 1]]((9x^2 - 1)^5) + (5x^4 - x)^3[5(9x^2 - 1)^4[18x]]](5x^5 + x) - ((5x^4 - x)^3(9x^2 - 1)^5)[25x^4 + 1]}{(5x^5 + x)^2}.$$

Here, the Quotient Rule is in red, the Product Rule is in blue, and the Chain Rule is in green (twice).

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