Section 2.21. Cauchy-Riemann Equations

Note. In this section, we introduce the Cauchy-Riemann equations, which are a necessary condition for the differentiability of a function f at a point z_0. In the next section, we show that these equations, along with some added continuity hypotheses, are sufficient for the differentiability of function f at point z_0.

Note. Based on the *Encyclopedia of Mathematics* website; accessed 1/3/2020), the Cauchy-Riemann equations "apparently occurred for the first time in the works of J. d'Alembert" in his *Essai d'une nouvelle théorie de la résistance des fluides*, Paris (1752). "Their first appearance as a criterion for analyticity was in a paper of L. Euler, delivered at the Petersburg Academy of Sciences in 1777," and appearing in *Nova Acta Acad. Sci. Petrop.*, **10**, 3–19 (1797). "A.L. Cauchy utilized the [equations] to construct the theory of functions, beginning with a memoir presented to the Paris Academy in 1814," in "Mémoire sur les intégrales définies," *Oeuvres complètes Ser. 1*, **1**, 319–506, Paris (1882). "The celebrated dissertation of B. Riemann on the fundamentals of function theory dates to 1851;" see "Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen komplexen Grösse," H. Weber (ed.), *Riemann's gesammelte math. Werke*, Dover Publications, 3–48 (1953, reprint).

Theorem 2.21.A. Differentiable Implies the Cauchy-Riemann Equations Suppose that $f(z) = u(x, y) + iv(x, y)$ and that f' exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x}[u(x, y)] = \frac{\partial}{\partial y}[v(x, y)] \text{ and } \frac{\partial}{\partial y}[u(x, y)] = -\frac{\partial}{\partial x}[v(x, y)]$$

(or with subscripts representing partial derivatives, $u_x = v_y$ and $u_y = -v_x$) at (x_0, y_0). Also, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Note. The proof is rather elementary and results from considering a limit as $\Delta z \to 0$ along the real axis and along the imaginary axis. We now present the proof.

Example 2.21.1. In Example 2.19.1, we saw that for $f(z) = z^2$ that $f'(z) = 2z$ and that this holds for all $z \in \mathbb{C}$. So f must satisfy the Cauchy-Riemann equations at all points (x, y). First, $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We then have $u_x = 2x = v_y$ and $u_y = -2y = -v_x$, so the Cauchy-Riemann equations are satisfied. We also have by Theorem 2.21.A that $f'(z) = u_x + iv_x = 2x + i(2y) = 2z$.

Example 2.21.2. Since the Cauchy-Riemann equations are necessary for (pointwise) differentiability, we can use them to find points where f is not differentiable. Consider $f(z) = |z|^2 = |x + iy|^2 = (x^2 + y^2) + i(0)$. We have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. So $u_x = 2x$, $u_y = 2y$, and $v_x = v_y = 0$. So for the Cauchy-Riemann equations to be satisfied, we need $x = 0 = y$. That is, the only point at which f *may* be differentiable is at $z_0 = 0 + 0i$. Once we see the sufficient conditions for differentiability, then we will see that $f(z) = |z|^2$ *is* in fact differentiable at $z_0 = 0$ and that $f'(0) = 0$.

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