Section 2.22. Sufficient Conditions for Differentiability

Note. In the previous section, we saw that if function f is differentiable point z_0 then the Cauchy-Riemann equations must be satisfied at z_0. In this section, we show that the Cauchy-Riemann equations are sufficient for the differentiability of function f at point z_0 with some added continuity conditions on the partial derivatives of $u(x, y)$ and $v(x, y)$.

Note. Consider the function

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0. \end{cases}$$

In Exercise 2.20.9 (also Exercise 2.20.9 in the 9th edition of the book) it is shown that $f'(0)$ does not exist. However, in Exercise 2.23.6 (Example 2.22.3 in the 9th edition fo the book) it is shown that f satisfies the Cauchy-Riemann equations at $z_0 = 0$. So the Cauchy-Riemann equations are not, alone, sufficient to guarantee the differentiability of f. We now give sufficient conditions for the differentiability of f at z_0.

Theorem 2.22.A. The Cauchy-Riemann Equations and Continuity Imply Differentiability

Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Then $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Example 2.22.1. Consider $f(x) = e^z = e^x e^{iy}$ (where $z = x + iy$). By Euler's formula, we have $f(z) = e^x \cos y + ie^x \sin y$, so $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Since $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$ and these derivatives are continuous for all (x, y) then, by Theorem 2.22.A, $f'(z) = u_x(x, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = e^z$ for all $z \in \mathbb{C}$. This is consistent with your ideas from Calculus 2 and will be verified again when we consider power series in Chapter 5.

Example 2.22.2. We saw in Example 2.21.2 that $f(z) = |z|^2$ satisfies the Cauchy Riemann equations at $z_0 = 0$. Since $f(z) = |z|^2 = (x^2 + y^2) + i(0)$, then $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ then the first-order partial derivatives of u and v exist and are continuous everywhere, so by Theorem 2.22.A, f is differentiable at $z_0 = 0$.

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