Section 2.23. Polar Coordinates

Note. In this section, we restate the results of the previous two sections on the Cauchy-Riemann equations, but this time in polar coordinates (r, θ) instead of rectangular coordinates (x, y).

Lemma 2.23.A. Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Then with $z_0 = r_0 \exp(i\theta_0) \neq 0$ we have

$$r_0 u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \text{ and } u_\theta(r_0, \theta_0) = -r_0 v_r(r_0, \theta_0).$$

These are the polar coordinate forms of the Cauchy-Riemann equations.

Lemma 2.23.B. Let $f(z) = f(r \exp(i\theta)) = u(r, \theta) + iv(r, \theta)$ be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$ and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form $ru_r = v_\theta$ and $u_\theta = -rv_r$ of the Cauchy Riemann equations at (r_0, θ_0).

Then the Cauchy-Riemann equations in rectangular form are satisfied at $z_0 = x_0 + iy_0$:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Note. The proof of Lemma 2.23.B is to be given in Exercise 2.23.7 (Exercise 2.24.5 in the 9th edition of the book).

Lemma 2.23.C. Let $f(z) = f(r\exp(i\theta)) = u(r, \theta) + iv(r, \theta)$ satisfy the hypotheses of Lemma 2.23.B. Then f is differentiable at $z_0 = r_0 \exp(i\theta_0)$ and

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)).$$

Note. The proof of Lemma 2.23.C is to be given in Exercise 2.23.8 (Exercise 2.24.6 in the 9th edition of the book). An alternative formula for $f'(z_0)$ is to be given in Exercise 2.23.9 (Exercise 2.24.7(a) in the 9th edition of the book):

$$f'(z_0) = \frac{-i}{z_0}(u_\theta(r_0, \theta_0) + iv_\theta(r_0, \theta_0)).$$

Note. Lemmas 2.23.A, 2.23.B, and 2.23.C combine to give the following.

Theorem 2.23.A. Let the function $f(z) = f(r\exp(i\theta)) = u(r,\theta) + iv(r,\theta)$ be defined throughout some ε neighborhood of a point $z_0 = r_0\exp(i\theta_0)$, and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood, and

(b) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form $ru_r = v_\theta$ and $u_\theta = -rv_r$ of the Cauchy-Riemann equations at (r_0, θ_0).

Then $f'(z_0) = e^{-i\theta_0}(u_r(r_0,\theta_0) + iv_r(r_0,\theta_0))$.

Example 2.23.2. Define $f(z) = f(r\exp(i\theta)) = \sqrt[3]{r}\exp(i\theta/3)$ for $r > 0$, $\alpha < \theta < \alpha + 2\pi$ for some fixed real α. This is "a cube root function." We have

$$u(r,\theta) = \sqrt[3]{r}\cos\left(\frac{\theta}{3}\right) \text{ and } v(r,\theta) = \sqrt[3]{r}\sin\left(\frac{\theta}{3}\right), \text{ so}$$

$$ru_r(r,\theta) = r\left(\frac{1}{3}r^{-2/3}\cos\left(\frac{\theta}{3}\right)\right) = \frac{r^{1/3}}{3}\cos\left(\frac{\theta}{3}\right) = v_\theta(r,\theta) \text{ and}$$

$$u_\theta(r,\theta) = -\frac{\sqrt[3]{r}}{3}\sin\left(\frac{\theta}{3}\right) = -r\left(\frac{1}{3}r^{-2/3}\sin\left(\frac{\theta}{3}\right)\right) = -rv_r.$$

So by Theorem 2.23.A, f is differentiable at all points at which it is defined and

$$f'(z) = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta)) = e^{-i\theta}\left(\frac{1}{3(\sqrt[3]{r})^2}\cos\left(\frac{\theta}{3}\right) + i\frac{1}{3(\sqrt[3]{r})^2}\sin\left(\frac{\theta}{3}\right)\right)$$

$$= \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2}e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2}.$$

Notice that this derivative is similar to what we would expect if we were to differentiate the real cube root function: $f(x) = x^{1/3}$ implies $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$. We cannot explore root functions in detail until after we introduce the exponential function e^z and complex logarithm functions.

Revised: 3/14/2020