Section 2.24. Analytic Functions

Note. In a sense, most of complex analysis is the study of analytic functions. There are a number of potential definitions of "analytic function," but whenever you hear this term, think "power series representation." In fact, this is the definition of "analytic" in the real setting (see my online notes for Analysis 2 [MATH 4227/5227] on 8-3. Taylor Series). In the complex setting, it is standard to have a different definition of analytic, but when we are done we will have the property that a function of a complex variable is analytic if and only if it has a power series representation (at a point or on a set). In this section, we give Brown and Churchill's definition of an analytic function of a complex variable and prove one result concerning analytic functions.

Definition. A function f of complex variable z is *analytic at a point* z_0 if it has a derivative at each point in some neighborhood of z_0. Function f is *analytic on set* S if f is analytic at every point of an open set containing set S. An *entire function* is analytic in the entire complex plane.

Note. The study of entire functions is an area of complex analysis. The classic text in this area is Ralph Boas' *Entire Functions*, NY: Academic Press (1954).

Note. In Example 2.19.3 we saw that $f(z) = |z|^2$ is only differentiable at $z_0 = 0$, so it is not analytic anywhere. Of course polynomials are examples of entire functions (as are $\sin z$, $\cos z$, and e^z, as we will see later).

Definition. If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is a *singular point* of f. That is, z_0 is a singular point if for all $\varepsilon > 0$ there is some $z_1 \in \mathbb{C}$ with $0 < |z_0 - z_1| < \varepsilon$ such that f is analytic at z_1.

Lemma 2.24.A. If f and g are analytic in a domain D, then

(i) for all $a, b \in \mathbb{C}$, $af(z) + bg(z)$ is analytic in D, and

(ii) if $g(z) \neq 0$ for $z \in D$, then $f(z)/g(z)$ is analytic in D.

Note. The proof of Lemma 2.24.A(i) is to be given in Exercise 2.25.4 (Exercise 2.26.3 in the 9th edition of the book). The proof of part (ii) similarly follows from the Quotient Rule.

Lemma 2.24.B. Suppose that f is analytic in a domain D and that g is analytic on the image of D under f, $f(D) = \{w \in \mathbb{C} \mid w = f(z) \text{ for some } z \in D\}$. Then the composition $g \circ f(z) = g(f(z))$ is analytic on D.

Note. To prove Lemma 2.24.B, we only need to use Chain Rule. A similar result is addressed for composition of entire functions in Exercise 2.25.4 (Exercise 2.26.3 in the 9th edition).

Theorem 2.24.A. If $f'(z) = 0$ everywhere in a domain D, then f must be constant throughout D.